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CONTINGENCY TABLES

A THESIS

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

CHAPTER I

INTRODUCTION

Analysis of contingency tables is one of the classical subjects in statistics. From the time of Karl Pearson's famous paper (PEARSON) in 1900 until today an increasing number of articles and papers have been written on the subject.

One of the purposes of this paper is to survey the literature and dispel some of the confusion which is often found regarding the fixing of marginal totals. Considerable care has been taken in an attempt to illustrate the application of the theory in each case.

A survey of the literature also reveals inadequate development of a theory of the power of statistical tests based on contingency tables. We have attempted to strengthen this area by investigating the power of some tests for 2 x 2 contingency tables with no marginal totals fixed.

A third purpose of the paper is to investigate the state of the limit theory for the χ^2 statistic and the relationship of the χ^2 and $-2 \log \lambda$ statistics.

In Chapter II the hypotheses to be tested are discussed. The cases involving marginal totals are also stated, and their effect upon the hypotheses is discussed.

Tests of the hypotheses are developed in Chapter III. Again, the distinction between the cases involving the marginal totals is carefully made. The application of the PEARSON-FISHER THEOREM, for which a rather general version due to BIRCH is given in Appendix A, is illustrated. The relationship between the χ^2 statistic and the $-2 \log \lambda$ statistic is also given in Chapter III.

Chapter IV is devoted to a discussion of the power of three tests of the independence hypothesis for 2×2 contingency tables with no marginal totals fixed.

For some points of theory no clear analytical development has been found in the literature. This will be mentioned as these points arise in the sequel.

In Chapter V some extensions of the theory and applications to 3-dimensional contingency tables are illustrated.

The notation to be used is illustrated by the contingency table

$$\begin{array}{cccc|c}
n_{11} & n_{12} & \cdots & n_{1s} & n_{1\cdot} \\
n_{21} & n_{22} & \cdots & n_{2s} & n_{2\cdot} \\
\vdots & \vdots & & \vdots & \vdots \\
n_{r1} & n_{r2} & & n_{rs} & n_{r\cdot} \\
\hline
n_{\cdot 1} & n_{\cdot 2} & \cdots & n_{\cdot s} & n
\end{array} .$$

That is, n_{ij} denotes the number of outcomes which belong to row classification i and column classification j. The total number of outcomes which belong to the i^{th} row classification is

$$n_{i\cdot} = \sum_j n_{ij} ,$$

and the total number of outcomes which belong to the j^{th} column classification is

$$n_{\circ j} = \sum_i n_{ij} \ .$$

The total number of observations is n, so

$$\sum n_{ij} = \sum n_{i\circ} = \sum n_{\cdot j} = n \ .$$

Similar notation is used for probabilities. Thus the probability to be associated with cell (i,j) is denoted p_{ij} and the row and column probabilities are denoted $p_{i\circ}$ and $p_{\cdot j}$, respectively.

The letter i will assume the values $1,2,\ldots,r$ and j will assume the values $1,2,\ldots,s$ throughout.

The notation $o(\ )$ and $O(\ )$ is used freely in the paper, especially in the proof in Appendix A. $F(t) = o(f(t))$ as $t \to a$ means that given ε there is a neighborhood N_ε of a such that $|F(t)| < \varepsilon\, |f(t)|$ for $t \in N_\varepsilon$, i.e. $\lim_{t \to a} \frac{F(t)}{f(t)} = 0$, while $G(t) = O(g(t))$ as $t \to a$ means that there is a constant M such that $|G(t)| \leq M|g(t)|$ in some neighborhood of a. The constant a can be real or vector valued, or ∞.

E_k denotes k-dimensional Euclidean space.

References are cited by capitalizing the name of the author. If more than one reference for a particular author is listed in the Bibliography, the number in brackets [ ] denotes the article.

CHAPTER II

HYPOTHESES AND SAMPLE SPACES

There are two hypotheses regarding contingency tables which are commonly tested. These hypotheses are usually called the "independence hypothesis" and the "homogeneity hypothesis."

We first discuss independence. For a general development of the subject see LOÉVE, p. 223.

Consider an abstract (finite) space, Ω, which is the cartesian product of two spaces, Ω_1 and Ω_2, so that $\Omega = \Omega_1 \times \Omega_2$. Let $\Omega_1 = \{\omega_{11}, \ldots, \omega_{1r}\}$ and $\Omega_2 = \{\omega_{21}, \ldots, \omega_{2s}\}$. Then $\Omega = \{(\omega_{1i}, \omega_{2j}) \mid i = 1, \ldots, r;\ j = 1, \ldots, s\}$.

Now assume a probability measure P on the sets of Ω such that

$$p_{ij} = P\left(\{(\omega_{1i}, \omega_{2j})\}\right)$$

for $i = 1, \ldots, r$ and $j = 1, \ldots, s$, where

$$\sum_{i,j} p_{ij} = 1 .$$

This measure P induces marginal measures P_1 and P_2 on the sets of Ω_1 and Ω_2, respectively, and we may define

$$p_{i.} = P_1\left(\{\omega_{1i}\}\right) = \sum_{j=1}^{s} P\left(\{(\omega_{1i}, \omega_{2j})\}\right) .$$

Thus, by virtue of the additivity property of a probability measure, we have

$$p_{i\circ} = P\left(\bigcup_j \{(\omega_{1i}, \omega_{2j})\}\right),$$

i.e. $p_{i.}$ is the accumulation of probability on the cylinder over ω_{1j}; and

$$p_{.j} = \sum_{i=1}^{r} P\left(\{(\omega_{1i}, \omega_{2j})\}\right) = P\left(\bigcup_i \{(\omega_{1i}, \omega_{2j})\}\right),$$

where $p_{\circ j}$ has an interpretation corresponding to that of $p_{i\circ}$.

Consider now the class C_1 of cylinder subsets of Ω of the form $\bigcup_j \{(\omega_{1i}, \omega_{2j})\}$ and the class C_2 of cylinder subsets of Ω of the form $\bigcup_i \{(\omega_{1i}, \omega_{2j})\}$. If these two classes have the property that, for any two subsets c_1 and c_2, one from each class

$$P(c_1 \cap c_2) = P(c_1)\, P(c_2),$$

then the two classes are said to be <u>independent</u>. This is what we shall mean by independence in a contingency table.

To obtain the usual form of the statement of the independence hypothesis, we note that if

$$c_i = \bigcup_j \{(\omega_{1i}, \omega_{2j})\}$$

and

$$c_j = \bigcup_i \{(\omega_{1i}, \omega_{2j})\},$$

then

$$c_i \cap c_j = \{(\omega_{1i}, \omega_{2j})\}$$

and the independence hypothesis is

$$p_{ij} = P(c_i)P(c_j) = P\left(\bigcup_j \{(\omega_{1i}, \omega_{2j})\}\right) P\left(\bigcup_i \{(\omega_{1i}, \omega_{2j})\}\right)$$

$$= \sum_i p_{ij} \sum_j p_{ij}$$

or

$$p_{ij} = p_{i.}\, p_{.j} \quad .$$

Thus our independence hypotheses asserts that, since row and column classifications are independent, the probability that a random observation will fall into cell (i,j) is the product of the probability that the element will fall into the i^{th} row classification and the probability that it will fall into the j^{th} column classification.

This hypothesis is sometimes called the hypothesis of "no interaction" between rows and columns, or "no association" between rows and columns.

The other hypothesis which is often tested is the hypothesis of homogeneity. In this hypothesis we consider the rows of an r x s contingency table as r samples, one from each of r populations, where the sample sizes are $n_{1.}, \ldots, n_{r.}$. Each member of a population may be classified into one of s categories, and n_{ij} is the number (in $n_{i.}$ random observations) from the i^{th} population which belong to the j^{th} category.

Let $p_j^{(i)}$ denote the probability that a randomly chosen element from the i^{th} population falls into the j^{th} category. The homogeneity hypothesis asserts that the r samples are from the same population, that is, that

$$p_j^{(i)} = p_j^{(k)}$$

for all $i = 1,\ldots,r$, $j = 1,\ldots,s$ and $k = 1,\ldots,r$. If we let p_{ij} be the probability that an element chosen randomly without regard to population belongs to the i^{th} population and the j^{th} category and if we use conditional probability, we have

$$p_j^{(i)} = \frac{p_{ij}}{p_{i\circ}} ,$$

so that this hypothesis may be stated mathematically as

$$p_{ij} = \frac{p_{i\circ}\, p_{kj}}{p_{k\circ}}$$

for $i = 1,\ldots,r$, $j = 1,\ldots,s$ and $k = 1,\ldots,r$.

The hypotheses of independence and homogeneity are equivalent, as we shall now show.

Suppose that $p_{ij} = p_{i\circ}p_{\circ j}$.

Then

$$p_{kj} = p_{k\circ}\, p_{\circ j}$$

and solving each equation for $p_{\circ j}$ yields

$$p_{ij} = \frac{p_{i\circ}\, p_{kj}}{p_{k\circ}} .$$

Thus independence implies homogeneity.

Conversely, if $p_{ij} = \frac{p_{i\cdot}p_{kj}}{p_{k\cdot}}$, then

$$p_{ij}p_{k\cdot} = p_{i\cdot}p_{kj}$$

so that

$$\sum_k p_{ij}p_{k\cdot} = \sum_k p_{i\cdot}p_{kj}$$

or

$$p_{ij} = p_{i\cdot}p_{\cdot j}$$

since

$$\sum_k p_{k\cdot} = 1 \quad \text{and} \quad \sum_k p_{kj} = p_{\cdot j} .$$

Hence homogeneity implies independence and the two hypotheses are equivalent.

In drawing a sample to use in testing one of these hypotheses there are three cases of interest involving the marginal totals of the contingency table. The totals may be completely unspecified, one set (row or column) may be specified or both sets may be specified, before drawing the sample.

On occasion the restrictions are imposed by conditions whose motivation is simply the pragmatic one of obtaining parameter-free critical regions. More will be said about this in Chapter III.

In the first case in which neither set of totals is fixed, n independent observations are made and each classified into a cell (i,j).

In the second case the row totals, for example, are fixed so that $n_{i\cdot}$ elements are chosen from the "i^{th} row" population and classified into the proper column category. Both sets of totals are fixed by experimental design in the third case.

These cases are sometimes referred to as the "double dichotomy," "comparative trial" and "independence trial," respectively. For further discussion see HARKNESS AND KATZ and BARNARD.

It is clear that the case in which one set of totals is fixed suggests testing the hypothesis of homogeneity, while the other two seem to suggest that the independence hypothesis is of interest.

Since fixing the row totals determines the probability that an observation will fall into a given row and similarly for the column probabilities, the statement of a hypothesis will depend upon the sampling procedure as follows:

Case I (no totals fixed).

Independence hypothesis: $p_{ij} = p_{i\cdot}p_{\cdot j}$.

Homogeneity hypothesis: $p_{ij} = \dfrac{p_{i\cdot}p_{kj}}{p_{k\cdot}}$.

Case II (one set of totals fixed). (row totals fixed here)

Independence hypothesis: $p_{ij} = \dfrac{n_{i\cdot}p_{\cdot j}}{n}$.

Homogeneity hypothesis: $p_{ij} = \dfrac{n_{i\cdot}p_{kj}}{n_{k\cdot}}$.

Case III (both sets of totals fixed).

Independence hypothesis: $p_{ij} = \dfrac{n_{i\cdot}n_{\cdot j}}{n^2}$.

$$\text{Homogeneity hypothesis:} \quad p_{ij} = \frac{n_{i\circ}p_{kj}}{n_{k\circ}}, \quad \text{where} \quad \sum_i p_{ij} = \frac{n_{\circ j}}{n} \,.$$

Note that the rows have been considered as samples from each of r populations for the homogeneity hypothesis. If the columns are to represent samples from each of s populations, then in Case I, for example, the hypothesis would be

$$p_{ij} = \frac{p_{\circ j}p_{ik}}{p_{\circ k}} \,.$$

Cases I and II seem quite natural. Case III may seem artificial but actually arises in practice. Suppose, for example, that 25 boys and 25 girls are given an automobile driving test and ranked in such a way that no "ties" are allowed. The results are then summarized in a 2 x 3 table in which the top ten ranks are called above average and the bottom ten below average. The marginal totals will then be fixed:

	above average	average	below average	total
boys				25
girls				25
total	10	30	10	50

CHAPTER III

SOME TESTS OF THE HYPOTHESES

In this chapter we discuss tests of the independence and homogeneity hypotheses. (Since the hypotheses are equivalent, their tests will be identical.) We begin by considering exact (discrete) probabilities, and later consider statistics whose distributions are asymptotic to the distribution of a continuous random variable.

Let T be the event that the table

$$\begin{matrix} n_{11} & n_{12} & \cdots & n_{1s} \\ n_{21} & n_{22} & \cdots & n_{2s} \\ \vdots & \vdots & & \vdots \\ n_{r1} & n_{r2} & \cdots & n_{rs} \end{matrix}$$

is observed, where $\sum n_{ij} = n$. Then a formula for $P(T)$ in terms of the n_{ij}'s will depend upon the model for sampling. Thus we consider again the three cases: Case I, no marginal totals fixed; Case II, one set of totals fixed; and Case III, both sets of marginal totals fixed.

Case III will be discussed first. In this case, the independence hypothesis is

$$p_{ij} = \left(\frac{n_{i\cdot}}{n}\right)\left(\frac{n_{\cdot j}}{n}\right) ,$$

where p_{ij} is the probability that a randomly chosen element falls into cell (i,j).

Consider a sequence of n events in which a_{ij} denotes the event that an observation falls into cell (i,j), and the number of times which a_{ij} occurs is n_{ij}. Then the population to be sampled in Case III is the set of all such sequences for which $\sum_j n_{ij} = n_{i.}$ and $\sum_i n_{ij} = n_{.j}$, where the $n_{i.}$'s and $n_{.j}$'s are fixed.

Let p be the probability of the occurrence of a particular sequence

$$a_{12},\ a_{31},\ a_{11},\ a_{13},\ldots,a_{11},\ a_{12}\ .$$

Then

$$p = P(a_{12})\ P(a_{31}|a_{12})\ P(a_{11}|a_{12},a_{31})\ldots P(a_{12}|a_{12},a_{31},a_{11},\ldots,a_{11})\ .$$

Reinterpreting the implication of the independence hypothesis for each factor of the expression on the right side, the right side becomes

$$\left[\frac{n_{1.}}{n}\ \frac{n_{.2}}{n}\right]\ \left[\frac{n_{3.}}{n-1}\ \frac{n_{.1}}{n-1}\right]\left[\frac{n_{1.}-1}{n-2}\ \frac{n_{.1}-1}{n-2}\right]\left[\frac{n_{1.}-2}{n-3}\ \frac{n_{.3}}{n-3}\right]\ldots[(\tfrac{2}{2})(\tfrac{1}{2})][(\tfrac{1}{1})(\tfrac{1}{1})].$$

Simplifying,

$$p = \frac{\Pi(n_{i.}!)\ \Pi(n_{.j}!)}{(n!)^2}$$

or

$$p = \frac{1}{\binom{n}{n_{1\cdot}, n_{2\cdot}, \ldots, n_{r\cdot}} \binom{n}{n_{\cdot 1}, n_{\cdot 2}, \ldots, n_{\cdot s}}} .$$

It is clear that p will have the same value for each sequence in the population. It can be verified directly (cf. KURTZ, p. 156) that there are

$$\binom{n}{n_{1\cdot}, n_{2\cdot}, \ldots, n_{r\cdot}} \binom{n}{n_{\cdot 1}, n_{\cdot 2}, \ldots, n_{\cdot s}}$$

such sequences. Then since T consists of $\binom{n}{n_{11}, n_{12}, \ldots, n_{rs}}$ of these sequences,

$$P(T) = \frac{\binom{n}{n_{11}, \ldots, n_{rs}}}{\binom{n}{n_{1\cdot}, \ldots, n_{r\cdot}} \binom{n}{n_{\cdot 1}, \ldots, n_{\cdot s}}}$$

(1) $$P(T) = \frac{\Pi(n_{i\cdot}!)\ \Pi(n_{\cdot j}!)}{n!\ \Pi(n_{ij}!)} .$$

Thus the hypotheses can be tested at rejection level α by choosing a rejection set R from the set of all tables having the fixed marginal totals so that

$$\sum_R P(T) = \alpha .$$

We follow the usual procedure of choosing for R as many of the least likely tables as possible so that

$$\sum_{R} P(T) \leq \alpha .$$

By choosing the rejection set R from the least likely tables we expect to obtain a more powerful test, in the sense of Neyman and Pearson, than we would obtain by choosing any other rejection set. A further discussion of power is in Chapter IV.

This test is called "Fisher's exact test" in honor of Sir Ronald A. Fisher, who developed it.

EXAMPLE. This example is inspired by Fisher's famous tea-testing experiment (FISHER [2]).

Ten martinis are mixed with identical proportions of gin and vermouth. Five are shaken and five are stirred. A person is asked to choose the five which are stirred.

We classify the results of this experiment into the four cells of the contingency table

		chosen		
		stirred	shaken	
actual	stirred			5
	shaken			5
		5	5	10

.

There are 5 possible tables having this set of marginal totals and their probabilities are calculated by using (1):

$$P(n_{11} = 0) = \frac{5!\ 5!\ 5!\ 5!}{10!\ 0!\ 5!\ 0!\ 5!} = 0.0040 .$$

$$P(n_{11} = 1) = 0.0992$$

$$P(n_{11} = 1) = 0.3968$$

$$P(n_{11} = 3) = 0.3968$$

$$P(n_{11} = 4) = 0.0992$$

$$P(n_{11} = 5) = 0.0040 \; .$$

Thus the rejection set for $\alpha = 0.05$ is

$$R = \left\{ n_{11} = 0, \quad n_{11} = 5 \right\} ,$$

and the actual rejection level is 0.0080.

Turning to Case II, we use the probabilities $p_j^{(i)}$ as before, and fix the row totals. Under the null hypothesis $p_j^{(i)} = p_j^{(k)}$, and we denote the common value by p_j. Then the probability of observing the i^{th} row is

$$\binom{n}{n_{i1}, \ldots, n_{is}} \prod_j p_j^{n_{ij}} ,$$

and, since the row samples are independent,

$$\begin{aligned} P(T) &= \prod_i \left[\binom{n_{i\cdot}}{n_{i1}, \ldots, n_{is}} \prod_j p_j^{n_{ij}} \right] \\ &= \prod_i \binom{n_{i\cdot}}{n_{i1}, \ldots, n_{is}} \prod_j p_j^{n_{\cdot j}} \; . \end{aligned}$$

We now have the problem that the p_j's are unspecified. As a means of obtaining a parameter-free distribution we calculate the probabilities, conditional upon fixed column totals, i.e. $P(T|M)$ where M

is the event that the column totals $n_{\circ 1}, \ldots, n_{\circ s}$ occur.

$$P(T|M) = \frac{\prod_i \binom{n_{i\circ}}{n_{i1}, \ldots, n_{is}} \prod_j p_j^{n_{\circ j}}}{\binom{n}{n_{\circ 1}, \ldots, n_{\circ s}} \prod_j p_j^{n_{\circ j}}}$$

so that

$$P(T|M) = \frac{\Pi(n_{i\circ}!)\ \Pi(n_{\circ j}!)}{n!\ \Pi(n_{ij}!)} \quad .$$

Thus the exact test again applies in Case II.

The exact test also applies in Case I, as we shall now show. We have

$$P(T) = \binom{n}{n_{11}, \ldots, n_{rs}} \prod_{i,j} p_{ij}^{n_{ij}}$$

which becomes under the independence hypothesis,

$$P(T) = \binom{n}{n_{11}, \ldots, n_{rs}} \prod_{i,j} p_{i\circ}^{n_{ij}}\ p_{\circ j}^{n_{ij}}$$

$$= \binom{n}{n_{11}, \ldots, n_{rs}} \prod_i p_{i\circ}^{n_{i\circ}} \ \prod_j p_{\circ j}^{n_{\circ j}}$$

Again we use the device of conditional probability to obtain a parameter-free distribution. In the calculation of $P(T|M)$, where M is the event that the observed set of marginal totals occurs, we need to find $P(M)$.

$$P(M) = \sum_M P(T)$$

$$= \sum_M \left[\binom{n}{n_{11}, \ldots, n_{rs}} \prod_i p_{i\circ}^{n_{i\circ}} \prod_j p_{\circ j}^{n_{\circ j}} \right]$$

$$= \left[\sum_M \binom{n}{n_{11}, \ldots, n_{rs}} \right] \prod_i p_{i\circ}^{n_{i\circ}} \prod_j p_{\circ j}^{n_{\circ j}} .$$

To obtain a simplification for

$$\sum_M \binom{n}{n_{11}, \ldots, n_{rs}},$$

consider the identity

$$\left(\sum_i x_i\right)^n \left(\sum_j y_j\right)^n = \left(\sum_{i,j} x_i y_j\right)^n .$$

By the multinomial theorem, this identity becomes

$$\left[\sum \left(\frac{n!}{n_{1\circ}! \ldots n_{r\circ}!} \prod_i x_i^{n_{i\circ}}\right)\right] \left[\sum \left(\frac{n!}{n_{\circ 1}! \ldots n_{\circ s}!} \prod_j y_j^{n_{\circ j}}\right)\right]$$

$$= \sum \left(\frac{n!}{n_{11}! \ldots n_{rs}!} \prod_{i,j} (x_i y_j)^{n_{ij}} \right)$$

$$= \sum \left(\frac{n!}{n_{11}! \ldots n_{rs}!} \prod \left[x_i^{(\sum_i n_{ij})} y_j^{(\sum_j n_{ij})} \right] \right)$$

where the summations are over all combinations of the $n_{i\circ}$'s, $n_{\circ j}$'s or

n_{ij}'s which are nonnegative and whose sum is n. Then by equating coefficients on the two sides, we have

$$\sum_{M} \frac{n!}{\Pi(n_{ij}!)} = \frac{n!}{\Pi(n_{i\cdot}!)} \; \frac{n!}{\Pi(n_{\cdot j}!)} \quad .$$

Hence

$$P(T|M) = \frac{\Pi(n_{i\cdot}!)\ \Pi(n_{\cdot j}!)}{n!\ \Pi(n_{ij}!)} ,$$

and the exact test again applies.

Therefore, Fisher's exact test is applicable to an $r \times s$ contingency table whether the test is based on an _a priori_ structure of fixed marginal totals or on an _a posteriori_ structure (conditional). The question arises as to the efficacy of the somewhat arbitrary model based upon conditioning. This question is resolved in the following argument, due to KURTZ, p. 171, in which the rejection sets from the conditional models are combined to obtain an unconditional test.

For a table of n elements there are a finite number of possible marginal totals. Let M_i denote a set of marginal totals, and denote

$$P(M_i) = q_i \ .$$

We wish to find a rejection set R such that

$$P(R) \leq \alpha ,$$

and if we use the exact test, R will be given by

$$R = \bigcup_i R_i ,$$

where R_i is the rejection set which results from applying the exact test when the marginal totals set is M_i.

Then

$$P(R_i | M_i) \leq \alpha \qquad \text{for each } i$$

and we wish to show that this implies that

$$P(R) \leq \alpha .$$

Since the R_i's are disjoint,

$$P(R) = \sum P(R_i)$$

$$= \sum P(R_i | M_i) q_i$$

$$\leq \sum \alpha \; q_i = \alpha .$$

We note that if $P(R|M_i) = \alpha$ for every i then $P(R) = \alpha$. Hence applying the exact test when the marginal totals are not fixed still results in a rejection level of α.

A more sophisticated treatment of this issue is given in MOOD AND GRAYBILL, p. 316.

Listing the possible tables and calculating their probabilities in order to apply the exact test can be quite tedious, especially if none of the marginal totals is small. In what follows we introduce two tests which do not require this listing of the unobserved tables, but which depend on limit theory for large samples and therefore carry some error in probability statements.

The test statistics for these two tests may also be used as a basis for the exact test. Used for this purpose, the test statistics provide an ordering for the sample tables so that the exact probabilities need to be computed only for those tables having a larger test statistic than the observed table. This procedure will be illustrated in the example at the end of this chapter.

CRAMÉR points out that in seeking a test of the independence hypothesis we wish to find a measure of the deviation of the sample distribution from the hypothetical distribution. In the case of no fixed totals (so that the hypothetical distribution is multinomial with unknown p_{ij}'s) such a measure is provided by

$$\sum c_{ij}\left(\frac{n_{ij}}{n} - p_{ij}\right)^2 ,$$

where p_{ij} is the probability that a random element of the population will fall into cell (i,j), and the c_{ij}'s are more or less arbitrarily chosen.

PEARSON of course had already discovered that, if $c_{ij} = \frac{n}{p_{ij}}$, a measure is obtained which has particularly simple properties. Let us consider

$$\sum \frac{n}{p_{ij}}\left(\frac{n_{ij}}{n} - p_{ij}\right)^2 = \sum \frac{(n_{ij} - np_{ij})^2}{np_{ij}} . \tag{2}$$

Here np_{ij} is the expected frequency in cell (i,j) in Case I, so that this expression may be written in the somewhat more mnemonic notation,

$$\sum \frac{(o_{ij} - e_{ij})^2}{e_{ij}} ,$$

where $o_{ij} = n_{ij}$ is the observed outcome and $e_{ij} = np_{ij}$ is the expected outcome for cell (i,j).

The p_{ij}'s are not known in general, but can be estimated from the sample values n_{ij}. The independence hypothesis is

$$p_{ij} = p_{i.}p_{.j}$$

and

$$\sum p_{i.} = \sum p_{.j} = 1 ,$$

so that we wish to estimate $r-1$ $p_{i.}$'s and $s-1$ $p_{.j}$'s or $r+s-2$ parameters.

The parameter space Θ can be taken to be an open subset of E_{r+s-2}. Using the principle of maximum likelihood estimation, we wish to find $\hat{\theta} = (\hat{p}_{1.},\ldots,\hat{p}_{.,s-1})$ which maximizes

$$\Pi(p_{i.}p_{.j})^{n_{ij}} ,$$

which is proportional to $P(T)$ under the independence hypothesis. Using logarithms, we may equivalently maximize

$$\sum n_{ij} \log p_{i.}p_{.j} = \sum_{\substack{i \neq r \\ j \neq s}} n_{ij} \log p_{i.}p_{.j} + \sum_{i=1}^{r-1} n_{is} \log p_{i.}(1 - \sum_{j=1}^{s-1} p_{.j}) +$$

$$+ \sum_{j=1}^{s-1} n_{rj} \log (1 - \sum_{i=1}^{r-1} p_{i.})p_{.j} + n_{rs} \log (1 - \sum_{i=1}^{r-1} p_{i.})(1 - \sum_{j=1}^{s-1} p_{.j}).$$

Taking partial derivatives with respect to the $p_{i\circ}$'s and $p_{\circ j}$'s and setting the resulting expressions equal to zero, we have

$$\frac{n_{i\circ}}{p_{i\circ}} - \frac{n_{r\circ}}{p_{r\circ}} = 0, \quad i = 1,\ldots,r-1$$

$$\frac{n_{\circ j}}{p_{\circ j}} - \frac{n_{\circ s}}{p_{\circ s}} = 0, \quad j = 1,\ldots,s-1 .$$

The solution to these equations, with the condition that

$$\sum p_{i\circ} = \sum p_{\circ j} = 1 ,$$

gives

$$\hat{p}_{i\circ} = \frac{n_{i\circ}}{n}, \qquad i = 1,\ldots,r$$

$$\hat{p}_{\circ j} = \frac{n_{\circ j}}{n}, \qquad j = 1,\ldots,s .$$

Then substitution in (2) yields

$$\text{(3)} \qquad \chi^2 = n \sum \frac{\left(n_{ij} - \frac{n_{i\circ} n_{\circ j}}{n}\right)^2}{n_{i\circ} n_{\circ j}} = n\left(\sum \frac{n_{ij}^2}{n_{i\circ} n_{\circ j}} - 1\right) ,$$

which is called the chi-square statistic. By the PEARSON-FISHER THEOREM, of which a fairly general version (due to BIRCH) is provided in Appendix A, the χ^2 statistic under the independence hypothesis is distributed asymptotically as a chi-square random variable with $rs - (r + s - 2) - 1 = (r - 1)(s - 1)$ degrees of freedom.

Thus to test the independence (or homogeneity) hypothesis when no totals are fixed we calculate the χ^2 statistic from observed sample values and compare it with the critical value in the chi-square distribution with $(r - 1)(s - 1)$ degrees of freedom, at rejection level α. That is, the observed χ^2 statistic is compared with the value for which the probability is α (chosen small) that a higher value of a chi-square variable occurs; if a chi-square value in excess of the critical value is observed, the hypothesis of independence (or homogeneity) is rejected on the principle that otherwise an unlikely event has occurred. The example at the end of this chapter illustrates the application of this test.

Another proof of a less general adaptation of the PEARSON-FISHER THEOREM is given in CRAMÉR. Cramér also proves the theorem for the special case in which the p_{ij}'s are known so that no parameters are estimated. For this special case, the statistic in (2) is asymptotically distributed as a chi-square variable with $rs - 1$ degrees of freedom.

To apply the above χ^2 test when one set of totals is fixed Cramér states (CRAMÉR, p. 446) that

$$\chi^2 = \sum_{i,j} \frac{(n_{ij} - n_{i.}\hat{p}_j)^2}{n_{i.}\hat{p}_j}$$

is asymptotically distributed as a chi-square variable with $r(s - 1) - t$ degrees of freedom, where t is the number of parameters to be estimated from the sample values, and the $\hat{p}_j$'s are estimates of the p_j parameters in Case II. He indicates the way in which his proof of the PEARSON-FISHER THEOREM can be altered to obtain this result.

We remark that under the homogeneity hypothesis

$$\sum_{j} \frac{(n_{ij} - n_{i\cdot}p_j)^2}{n_{i\cdot}p_j}$$

is asymptotically distributed as a chi-square variable with $s-1$ degrees of freedom, when the p_j's are known (CRAMÉR, p. 417). Moreover, since the row samples are independent,

$$\sum_{i,j} \frac{(n_{ij} - n_{i\cdot}p_j)^2}{n_{i\cdot}p_j}$$

is asymptotically distributed as a chi-square variable with $r(s-1)$ degrees of freedom when the p_j's are known. In effect Cramér's generalization shows that estimating t parameters reduces the number of degrees of freedom by t, as would be expected. Since there are $s-1$ p_j's to be estimated, the χ^2 statistic is asymptotically distributed as a chi-square variable with $r(s-1) - (s-1) = (r-1)(s-1)$ degrees of freedom, as in Case I.

The maximum likelihood estimates in this case are $\hat{p}_j = \frac{n_{\cdot j}}{n}$, so that the formula for calculating the χ^2 statistic above is the same as in the previous case.

No proof of the application of the chi-square distribution to Case III, in which both sets of marginal totals are fixed, has been found in the literature. The proofs given by Birch and Cramér are not valid for this case because of the independence requirement.

Keeping (KEEPING, p. 315) has stated the general result that placing k linear constraints upon the n_{ij}'s reduces the number of degrees of freedom of the asymptotic chi-square distribution of χ^2 by k. (Note that in Case I there is one linear constraint: $\sum n_{ij} = n.$)

In Case III there are rs probabilities and no parameters to be estimated, since the independence hypothesis,

$$p_{ij} = \frac{n_{i.}}{n}\frac{n_{.j}}{n},$$

completely specifies the p_{ij}'s. However, there are $r+s-1$ linear constraints:

$$\sum_i n_{ij} = n_{.j} \quad \text{for} \quad j = 1,\ldots,s$$

$$\sum_i n_{ij} = n_{i.} \quad \text{for} \quad i = 1,\ldots,r-1 .$$

Hence the χ^2 statistic is asymptotically distributed as a chi-square variable with $rs - (r + s - 1) = (r - 1)(s - 1)$ degrees of freedom, if Keeping's statement is correct. We also have in this case

$$\chi^2 = \sum \frac{\left(n_{ij} - \frac{n_{i.}n_{.j}}{n}\right)^2}{\frac{n_{i.}n_{.j}}{n}}$$

just as in the other cases.

The final test we shall consider is based on the asymptotic distribution, under the hypothesis of independence, of the statistic $-2 \log \lambda$. The statistic λ is the likelihood ratio:

$$\lambda = \frac{\sup_{\omega} P(T)}{\sup_{\Omega} P(T)} ,$$

where ω is the parameter space under the independence hypothesis and Ω is the complete parameter space.

If no totals are fixed, then

$$P(T) = \binom{n}{n_{11},\ldots,n_{rs}} \Pi \, p_{ij}^{n_{ij}} ,$$

and the maximum value of $P(T)$ occurs when

$$p_{ij} = \hat{p}_{ij} = \frac{n_{ij}}{n} .$$

Under the null hypothesis $p_{ij} = p_{i\cdot}p_{\cdot j}$,

$$P(T) = \binom{n}{n_{11},\ldots,n_{rs}} \prod_i p_{i\cdot}^{n_{i\cdot}} \prod_j p_{\cdot j}^{n_{\cdot j}}$$

and the maximum value of $P(T)$ occurs when

$$p_{i\cdot} = \hat{p}_{i\cdot} = \frac{n_{i\cdot}}{n} \quad \text{and} \quad p_{\cdot j} = \hat{p}_{\cdot j} = \frac{n_{\cdot j}}{n} .$$

Then the likelihood ratio in this case becomes

$$\lambda = \frac{\binom{n}{n_{11},\ldots,n_{rs}} \prod_i \left(\frac{n_{i\cdot}}{n}\right)^{n_{i\cdot}} \prod_j \left(\frac{n_{\cdot j}}{n}\right)^{n_{\cdot j}}}{\binom{n}{n_{11},\ldots,n_{rs}} \prod_{i,j} \left(\frac{n_{ij}}{n}\right)^{n_{ij}}}$$

which reduces to

$$\lambda = \frac{\prod_i n_{i\cdot}^{n_{i\cdot}} \prod_j n_{\cdot j}^{n_{\cdot j}}}{n^n \prod_{i,j} n_{ij}^{n_{ij}}}$$

We wish to find the asymptotic distribution of $-2 \log \lambda$.

$$(4) \qquad -2 \log \lambda = 2\left(\sum_{i,j} n_{ij} \log n_{ij} + n \log n - \sum_i n_{i\cdot} \log n_{i\cdot} - \sum_j n_{\cdot j} \log n_{\cdot j}\right)$$

$$= 2\left(\sum_{i,j} n_{ij} \log n_{ij} + \sum_{i,j} n_{ij} \log n - \sum_i \sum_j n_{ij} \log n_{i\cdot} - \sum_j \sum_i n_{ij} \log n_{\cdot j}\right)$$

$$= 2 \sum_{i,j} n_{ij} \log \frac{n_{ij}}{\frac{n_{i\cdot} n_{\cdot j}}{n}} .$$

Now let $x_{ij} = \dfrac{n_{ij} - \frac{n_{i\cdot} n_{\cdot j}}{n}}{\sqrt{n}}$, so that

$$n_{ij} = \sqrt{n}\ x_{ij} + \frac{n_{i\cdot} n_{\cdot j}}{n} .$$

Then

$$\log \frac{n_{ij}}{\frac{n_{i\cdot} n_{\cdot j}}{n}} = \log\left(1 + \frac{\sqrt{n}\, x_{ij}}{\frac{n_{i\cdot} n_{\cdot j}}{n}}\right)$$

$$= \frac{\sqrt{n}\ x_{ij}}{\frac{n_{i\cdot} n_{\cdot j}}{n}} - \frac{n\ x_{ij}^2}{2\left(\frac{n_{i\cdot} n_{\cdot j}}{n}\right)^2} + \frac{n^{3/2}\ x_{ij}^3}{3\left(\frac{n_{i\cdot} n_{\cdot j}}{n}\right)^3} + \dots ,$$

and

$$n_{ij} \log \frac{n_{ij}}{\frac{n_{i\cdot}n_{\cdot j}}{n}} = \sqrt{n}\ x_{ij} + \frac{nx_{ij}^2}{\frac{n_{i\cdot}n_{\cdot j}}{n}} - \frac{nx_{ij}^2}{2\left(\frac{n_{i\cdot}n_{\cdot j}}{n}\right)} - \frac{x_{ij}^3}{2\left(\frac{n_{i\cdot}n_{\cdot j}}{n^2}\right)^2} + \frac{x_{ij}^3}{3\left(\frac{n_{i\cdot}n_{\cdot j}}{n^2}\right)^2} + \cdots .$$

It should now be observed that

$$\sum_{i,j} \sqrt{n}\ x_{ij} = 0$$

and

$$\sum_{i,j} \frac{nx_{ij}^2}{\frac{n_{i\cdot}n_{\cdot j}}{n}} = \chi^2 .$$

Furthermore, omitting the details, the remaining terms of the series are $o(\frac{1}{\sqrt{n}})$, and it follows that

$$-2 \log \lambda = \chi^2 + o\left(\frac{1}{\sqrt{n}}\right) .$$

Finally, it can now be shown (cf. Lemma 13 and Lemma 14, Appendix A) that the distributions of $-2 \log \lambda$ and χ^2 are asymptotically equivalent. Hence $-2 \log \lambda$ is asymptotically distributed as a chi-square random variable with $(r-1)(s-1)$ degrees of freedom.

(For a direct approach to the asymptotic distribution of $-2 \log \lambda$, see WILKS. Using a different approach, essentially that of maximizing entropy, KULLBACK arrives at the same test statistic, $-2 \log \lambda$.)

Therefore, the independence or homogeneity hypothesis can be tested by using $-2 \log \lambda$ instead of the χ^2 statistic, with no theoretical superiority of either one. Using $-2 \log \lambda$ is often more convenient than using χ^2, especially if a table of values for $k \log k$ is available. Such a table may be found in NICHOLSON, for example.

We note that $k \log k$ must be given the value 0 for $k = 0$ in the calculation of $-2 \log \lambda$.

EXAMPLE. Let the observed marginal totals for a 2 x 3 contingency table be $n_{1\circ} = 12$, $n_{2\circ} = 12$, $n_{\circ 1} = 4$, $n_{\circ 2} = 8$ and $n_{\circ 3} = 12$. Then Figure 1 shows all possible tables having these totals. The number in parentheses below each table is the χ^2 statistic for that table, calculated from (3). Similarly, the number in brackets and the number below the one in brackets are the $-2 \log \lambda$ statistic from (4) and the exact (conditional) probability from (1), respectively.

To apply the exact test we determine the rejection set by adding up the exact probabilities, beginning with those tables having probability 0.0000 and proceeding in increasing order, until we have a total less than or equal to the rejection level α while using the maximum number of tables. Thus for $\alpha = 0.10$, the rejection set is the set of tables not enclosed by the solid lines. If the observed table belongs to this set, then the independence or homogeneity hypothesis is rejected.

To test the hypothesis of independence or homogeneity using only the χ^2 or $-2 \log \lambda$ statistic for the observed table, we must first find the critical value for a chi-square random variable with two degrees

0 0 12 4 8 0 (24.00) [33.27] 0.0000	1 0 11 3 8 1 (17.33) [20.89] 0.0000	2 0 10 2 8 2 (13.33) [16.84] 0.0001	3 0 9 1 8 3 (12.00) [15.28] 0.0003	4 0 8 0 8 4 (13.33) [17.99] 0.0002
0 1 11 4 7 1 (16.83) [20.36] 0.0000	1 1 10 3 7 2 (10.83) [11.93] 0.0008	2 1 9 2 7 3 (7.50) [8.13] 0.0039	3 1 8 1 7 4 (6.83) [7.47] 0.0059	4 1 7 0 7 5 (8.83) [10.94] 0.0023
0 2 10 4 6 2 (11.33) [13.46] 0.0007	1 2 9 3 6 3 (6.00) [6.28] 0.0091	2 2 8 2 6 4 (3.33) [3.45] 0.0301	3 2 7 1 6 5 (3.33) [3.47] 0.0328	4 2 6 0 6 6 (6.00) [7.64] 0.0096
0 3 9 4 5 3 (7.50) [9.19] 0.0046	1 3 8 3 5 4 (2.83) [2.91] 0.0411	2 3 7 2 5 5 (0.83) [0.84] 0.0985	3 3 6 1 5 6 (1.50) [1.55] 0.0766	4 3 5 0 5 7 (4.83) [6.39] 0.0164
0 4 8 4 4 4 (5.33) [6.90] 0.0128	1 4 7 3 4 5 (1.33) [1.38] 0.0822	2 4 6 2 4 6 (0.00) [0.00] 0.1438	3 4 5 1 4 7 (1.33) [1.38] 0.0822	4 4 4 0 4 8 (5.33) [6.90] 0.0128
0 5 7 4 3 5 (4.83) [6.39] 0.0164	1 5 6 3 3 6 (1.50) [1.55] 0.0766	2 5 5 2 3 7 (0.83) [0.84] 0.0985	3 5 4 1 3 8 (2.83) [2.91] 0.0411	4 5 3 0 3 9 (7.50) [9.19] 0.0046
0 6 6 4 2 6 (6.00) [7.64] 0.0096	1 6 5 3 2 7 (3.33) [3.47] 0.0328	2 6 4 2 2 8 (3.33) [3.45] 0.0301	3 6 3 1 2 9 (6.00) [6.28] 0.0091	4 6 2 0 2 10 (11.33) [13.46] 0.0007
0 7 5 4 1 7 (8.83) [10.94] 0.0023	1 7 4 3 1 8 (6.83) [7.47] 0.0059	2 7 3 2 1 9 (7.50) [8.13] 0.0039	3 7 2 1 1 10 (10.83) [11.93] 0.0008	4 7 1 0 1 11 (16.83) [20.36] 0.0000
0 8 4 4 0 8 (13.33) [17.99] 0.0002	1 8 3 3 0 9 (12.00) [15.28] 0.0003	2 8 2 2 0 10 (13.33) [16.84] 0.0001	3 8 1 1 0 11 (17.33) [20.89] 0.0000	4 8 0 0 0 12 (24.00) [33.27] 0.0000

Figure 1. Contingency Tables with Fixed Marginal Totals.

of freedom. The rejection set in this case is then the set of all tables for which the χ^2 (or $-2 \log \lambda$) statistic exceeds this critical value. For $\alpha = 0.10$ this value is 4.61. Thus using either of the two approximate tests leads to a larger rejection set in this example, as the occurrence of the tables set off by a broken line would lead to rejection using the approximate tests, but would not lead to rejection using the exact tests.

We also note that the two approximate tests do not always give the same results. For example, suppose that the observed table is

$$\begin{array}{ccc} 4 & 4 & 4 \\ 0 & 4 & 8 \end{array} .$$

Then at rejection level $\alpha = 0.05$, the critical value is 5.99 so that if $-2 \log \lambda$ is used the independence hypothesis is rejected, while, if χ^2 is used, the hypothesis is not rejected. For the table in Figure 1 it happens that $-2 \log \lambda \geq \chi^2$, but this is not true in general.

To illustrate a previous remark concerning the use of the χ^2 or $-2 \log \lambda$ statistics to rank the possible tables before applying the exact test, suppose that

$$\begin{array}{ccc} 0 & 2 & 10 \\ 4 & 6 & 2 \end{array}$$

is the observed table. We now calculate χ^2 or $-2 \log \lambda$ for every table having the same marginal totals. Note that this is much easier than calculating the exact probability for every such table. Now calculate the exact probability for each table, beginning with the table having the largest χ^2 (or $-2 \log \lambda$) statistic and proceeding in decreasing order of the statistics. If the sum of the exact probabilities exceeds α before or when the observed table is reached, the hypothesis is not rejected. Otherwise the hypothesis is rejected.

For this observation, $-2 \log \lambda = 13.46$. The sum of the exact probabilities for tables having $-2 \log \lambda$ statistic greater than or equal to 13.46 is 0.0026. Hence for this observation, we would reject the hypothesis if $\alpha \geq 0.0026$.

It is generally agreed that the approximate tests are not satisfactory for 2×2 tables unless n is rather large. Of course, the χ^2 or $-2 \log \lambda$ statistic can still be used to rank the possible tables having the same marginal totals to aid in applying the exact test.

Yates (YATES) has recommended that, for 2×2 tables, a "correction for continuity" in the χ^2 statistic be used. The Yate's formula for the χ^2 statistic for 2×2 tables is

$$\chi^2 = n \sum \frac{\left(\left| n_{ij} - \frac{n_{i.}n_{.j}}{n} \right| - \frac{1}{2} \right)^2}{n_{i.}n_{.j}} .$$

Several tables are available which simplify the testing of the independence hypothesis for 2×2 contingency tables. Some examples are found in the following: FISHER AND YATES; ARMSEN; MAINLAND.

It should also be mentioned that

$$\frac{\chi^2}{n(q-1)}$$

where $q = \min\{r,s\}$ is a measure of the <u>degree</u> <u>of</u> <u>association</u> between row and column classifications (cf. CRAMÉR, p. 443). It can be shown that

$$0 \leq \frac{\chi^2}{n(q - 1)} \leq 1 \quad .$$

For further discussion of the application of the three tests, see COCHRAN and BARNARD. A fairly extensive list of other references not explicitly cited here is given in the Bibliography.

CHAPTER IV

POWER OF THE TESTS FOR CASE I

2 x 2 TABLES

Comparatively little is found in the literature pertaining to the power of tests of the hypothesis of independence or homogeneity in contingency tables. Some power functions have been considered for Case II tables (one set of marginal totals fixed) by PATNAIK and by SILLITTO; BENNETT AND HSU have calculated some exact powers for Case II and Case III tables; and HARKNESS AND KATZ have derived theoretical expressions for the power for Case I.

In each of these references, only 2 x 2 tables are considered. All but Harkness and Katz use the exact test, and Harkness and Katz use the uniformly most powerful unbiased test (UMPUT) of size α. The UMPUT is essentially the exact test with randomization in order to increase the probability of the rejection set to exactly α.

In this chapter emphasis is placed on the calculation of the exact power of the three previously discussed tests of the independence hypothesis for 2 x 2 contingency tables when no marginal totals are fixed.

By the power of a test we mean the probability that the hypothesis will be rejected. If the hypothesis is false, then the power is the probability of rejecting the false hypothesis, that is the probability that the type II error, in the sense of Neyman and Pearson, will not be made. If the hypothesis is true, then the power is the probability of the rejection set, which, under ideal conditions such as the aforementioned UMPUT, is α.

To compute the power the rejection set R must first be determined. This is done by applying the test to every possible outcome. We consider here the power for different values of n, so that for a given n, the test must be applied to every table for which $\sum n_{ij} = n$, in order to determine the rejection set.

Having determined R, the power is given by

$$\sum_{R} \frac{n!}{n_{11}!\, n_{12}!\, n_{21}!\, n_{22}!}\, p_{11}^{n_{11}}\, p_{12}^{n_{12}}\, p_{21}^{n_{21}}\, p_{22}^{n_{22}} \quad ,$$

where values must be assigned to the p_{ij}'s before the power can be computed. It is convenient to assign values to p_{11}, $p_{1.}$ and $p_{.1}$, and that is what is done here.

The power was computed, using a Burroughs B5000 computer, for the exact test and for each of the two approximate tests of the independence hypothesis. Several values of n and several values of p_{11}, $p_{1.}$ and $p_{.1}$ were used. Figure 2 shows some graphs made using the results of the computations.

The graphs in Figure 2 show, for example, that the probability of discovering a 0.1 difference between p_{11} and $p_{1.}p_{.1}$ varies from 0.242 when $n = 10$ to 0.985 when $n = 100$. It is sufficient to consider only the difference between p_{11} and $p_{1.}p_{.1}$ in a 2 x 2 table, for if

$$|p_{11} - p_{1.}p_{.1}| = k$$

then

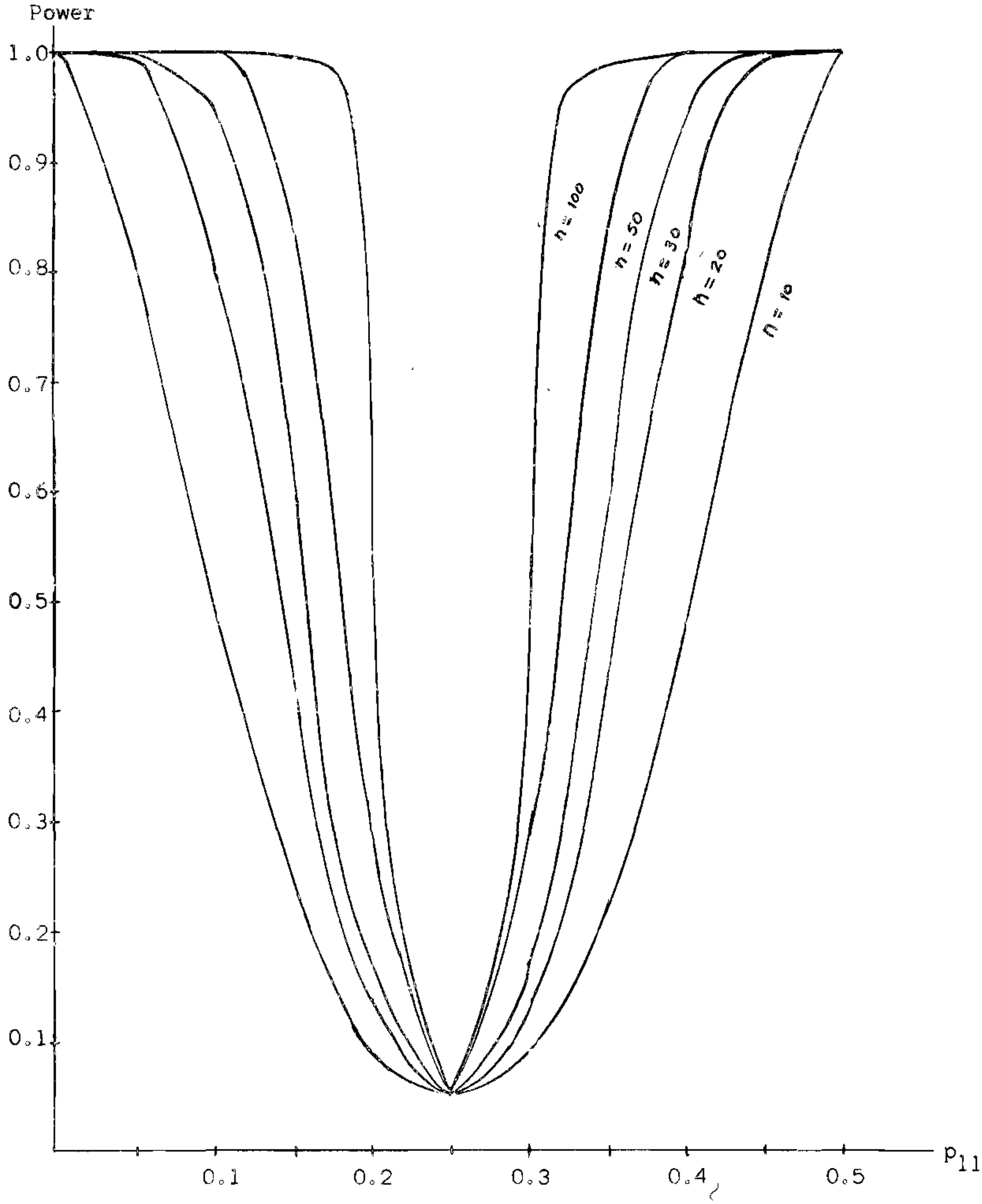

Figure 2. Power Curves for the χ^2 Statistic Test with $\alpha = 0.05$ and $p_{1.} = p_{.1} = 0.5$.

$$|p_{12} - p_{1\circ}p_{\circ 2}| = |(p_{1\circ} - p_{11}) - p_{1\circ}(1 - p_{\circ 1})|$$

$$= |p_{1\circ}p_{\circ 1} - p_{11}|$$

$$= k \quad ,$$

and similarly

$$|p_{ij} - p_{i\circ}p_{\circ j}| = k \quad \text{for} \quad i, j = 1,2 \ .$$

One question of interest is the question as to which of the three tests is most powerful.

It is known that the rejection set for the exact test is a subset of each of the rejection sets for the asymptotic tests, so that the exact test cannot be the most powerful. Thus there remains the question: "Is the test using the χ^2 statistic or the test using the $- 2 \log \lambda$ statistic more powerful?".

Theoretically, neither test should be consistently more powerful than the other, and the results in Table 1 demonstrate that this is the case. While the power using $- 2 \log \lambda$ is usually slightly higher, it is not always higher than the power using χ^2.

It should be noted here that the χ^2 statistic was calculated using (3) so that Yate's continuity correction was not included. Including the correction term could make some difference for small values of n, but should make no appreciable difference for large values of n.

Precaution must be taken in using the power graphs, since the power depends upon the marginal probabilities, not merely upon the deviation of p_{ij} from $p_{i\circ}p_{\circ j}$.

Table 1. Power for n = 30. IP11 is $p_{1.}p_{.1}$. UPX, UPL, and UPE denote the power when $\alpha = 0.05$ using the χ^2 statistic, $-2 \log \lambda$ statistic and exact tests, respectively. The suffix "U" denotes the corresponding power for $\alpha = 0.10$.

P1.	P.1	IP11	P11	UPX	UPL	UPE	UPXU	UPLU	UPEU
0.20	0.20	0.04	0.00	0.07337094	0.26184223	0.04290048	0.22840811	0.49700073	0.14625576
0.20	0.20	0.04	0.05	0.07608919	0.07578126	0.00738730	0.13171331	0.14379972	0.01473055
0.20	0.20	0.04	0.10	0.53106242	0.49873572	0.08664954	0.63328092	0.60356083	0.11504959
0.20	0.20	0.04	0.15	0.94304698	0.93056425	0.43517577	0.96468259	0.95531096	0.50124628
0.20	0.20	0.04	0.20	0.99876206	0.99876206	0.99876206	0.99876206	0.99876206	0.99876206
0.20	0.30	0.06	0.00	0.33017127	0.64282775	0.24025367	0.59017000	0.82783888	0.46342861
0.20	0.30	0.06	0.05	0.03853182	0.08630803	0.01582193	0.10160618	0.16038288	0.04818630
0.20	0.30	0.06	0.10	0.24129153	0.23455374	0.04633339	0.35095019	0.34122144	0.07624652
0.20	0.30	0.06	0.15	0.76394106	0.75057601	0.31303770	0.84778810	0.83546922	0.40050665
0.20	0.30	0.06	0.20	0.99720329	0.99411906	0.98500547	0.99855010	0.99855225	0.99144874
0.20	0.40	0.08	0.00	0.67416626	0.87885595	0.55067221	0.85506494	0.95303454	0.73706667
0.20	0.40	0.08	0.05	0.10752860	0.17955114	0.06870110	0.20870383	0.26937314	0.13705651
0.20	0.40	0.08	0.10	0.09248446	0.10697568	0.03131139	0.16915171	0.18604297	0.06339198
0.20	0.40	0.08	0.15	0.51040705	0.52289454	0.27454615	0.64541706	0.65183820	0.38657828
0.20	0.40	0.08	0.20	0.97811918	0.98740898	0.94383289	0.99340463	0.99451659	0.97233901
0.20	0.50	0.10	0.00	0.89648744	0.96364739	0.80599750	0.96639017	0.98559732	0.90148597
0.20	0.50	0.10	0.05	0.27475628	0.32183605	0.17357420	0.40812876	0.44404894	0.26967721
0.20	0.50	0.10	0.10	0.05098175	0.07292309	0.02276668	0.10991743	0.13480357	0.05109265
0.20	0.50	0.10	0.15	0.27475628	0.32183605	0.17117368	0.40812876	0.44404894	0.27599217
0.20	0.50	0.10	0.20	0.89648743	0.96364739	0.82431209	0.96639017	0.98559732	0.90148760
0.20	0.60	0.12	0.00	0.97811918	0.98740898	0.94057067	0.99340463	0.99451659	0.97223929
0.20	0.60	0.12	0.05	0.51040705	0.52289454	0.27924619	0.64541706	0.65183820	0.38436422
0.20	0.60	0.12	0.10	0.09248446	0.10697568	0.03194354	0.16915171	0.18604297	0.06223918
0.20	0.60	0.12	0.15	0.10752860	0.17955114	0.06860431	0.20870383	0.26937314	0.13958002
0.20	0.60	0.12	0.20	0.67416626	0.87885595	0.56592935	0.85506494	0.95303454	0.73706668
0.20	0.70	0.14	0.00	0.99720329	0.99411906	0.98498939	0.99855010	0.99855225	0.98984799
0.20	0.70	0.14	0.05	0.76394106	0.75057601	0.31406485	0.84778810	0.83546922	0.40037027
0.20	0.70	0.14	0.10	0.24129153	0.23455374	0.04683439	0.35095019	0.34122144	0.07615763
0.20	0.70	0.14	0.15	0.03853182	0.08630803	0.01581549	0.10160618	0.16038288	0.04830307
0.20	0.70	0.14	0.20	0.33017127	0.64282775	0.24262725	0.59017000	0.82783888	0.46342861
0.20	0.80	0.16	0.00	0.99876206	0.99876206	0.99876206	0.99876206	0.99876206	0.99876206
0.20	0.80	0.16	0.05	0.94304698	0.93056425	0.43517881	0.96468259	0.95531096	0.49931451
0.20	0.80	0.16	0.10	0.53106242	0.49873572	0.08667436	0.63328092	0.60356083	0.11427516
0.20	0.80	0.16	0.15	0.07608919	0.07578126	0.00738874	0.13171331	0.14379972	0.01460582
0.20	0.80	0.16	0.20	0.07337094	0.26184223	0.04294076	0.22840811	0.49700073	0.14625576
0.30	0.30	0.09	0.00	0.76611304	0.93738192	0.67769280	0.91898731	0.98245404	0.85945704
0.30	0.30	0.09	0.05	0.14850982	0.23660254	0.11651002	0.27819121	0.32795968	0.22582087
0.30	0.30	0.09	0.10	0.06338499	0.07500670	0.01356736	0.12846927	0.13928963	0.03499586
0.30	0.30	0.09	0.15	0.35821881	0.34747083	0.07578971	0.48571377	0.47719951	0.12071720
0.30	0.30	0.09	0.20	0.82764818	0.81640548	0.32364653	0.89597254	0.88781312	0.41478328
0.30	0.30	0.09	0.25	0.99320809	0.99163587	0.70812862	0.99678917	0.99590854	0.78927374
0.30	0.30	0.09	0.30	0.99997746	0.99997746	0.99997746	0.99997746	0.99997746	0.99997746
0.30	0.40	0.12	0.00	0.96369717	0.99419164	0.93175222	0.99223857	0.99869609	0.97736271
0.30	0.40	0.12	0.05	0.40035825	0.46340279	0.34011936	0.55620602	0.58150442	0.48062842
0.30	0.40	0.12	0.10	0.07233014	0.09580090	0.04953535	0.14571244	0.15984846	0.09866289
0.30	0.40	0.12	0.15	0.11894729	0.12220272	0.03782659	0.20652570	0.21008116	0.07206156
0.30	0.40	0.12	0.20	0.50246305	0.50211641	0.23256307	0.63945884	0.63874498	0.32774804
0.30	0.40	0.12	0.25	0.91463792	0.91401015	0.65110996	0.95773838	0.95665946	0.74236062
0.30	0.40	0.12	0.30	0.99988276	0.99979863	0.99924917	0.99996372	0.99996416	0.99967324
0.30	0.50	0.15	0.00	0.99732774	0.99934427	0.99162216	0.99954332	0.99977254	0.99727450
0.30	0.50	0.15	0.05	0.70166905	0.71979685	0.57332404	0.81481543	0.82222518	0.68943815
0.30	0.50	0.15	0.10	0.22037785	0.23650574	0.14392849	0.34037208	0.34936735	0.22517845
0.30	0.50	0.15	0.15	0.05237979	0.06055981	0.02612961	0.11079690	0.11734350	0.05575316
0.30	0.50	0.15	0.20	0.22037785	0.23650574	0.13318430	0.34037208	0.34936735	0.21608720
0.30	0.50	0.15	0.25	0.70166905	0.71979685	0.55552118	0.81481543	0.82222518	0.68111395
0.30	0.50	0.15	0.30	0.99732774	0.99934427	0.99269407	0.99954332	0.99977254	0.99727559
0.40	0.40	0.16	0.00	0.99920431	0.99993029	0.99763472	0.99991739	0.99998676	0.99948050
0.40	0.40	0.16	0.05	0.74810344	0.77627039	0.68836659	0.85351701	0.86043425	0.79731474
0.40	0.40	0.16	0.10	0.24480490	0.29597196	0.21869751	0.40009546	0.41003317	0.32976372
0.40	0.40	0.16	0.15	0.05425509	0.06458955	0.03396653	0.11626316	0.12079261	0.07069272
0.40	0.40	0.16	0.20	0.15033792	0.15504918	0.05874518	0.25096971	0.25199251	0.10820587
0.40	0.40	0.16	0.25	0.54217018	0.54659392	0.28713815	0.67919041	0.67876973	0.40055230
0.40	0.40	0.16	0.30	0.91886170	0.91887231	0.67341980	0.96081653	0.96016884	0.76184019
0.40	0.40	0.16	0.35	0.99885593	0.99871974	0.92270589	0.99964271	0.99958099	0.95266685
0.40	0.40	0.16	0.40	0.99999978	0.99999978	0.99999978	0.99999978	0.99999978	0.99999978
0.40	0.50	0.20	0.00	0.99999576	0.99999633	0.99997024	0.99999929	0.99999934	0.99999114
0.40	0.50	0.20	0.05	0.95319022	0.95740125	0.87517677	0.98051294	0.98079899	0.92413563
0.40	0.50	0.20	0.10	0.61608647	0.63402403	0.48069832	0.74880321	0.75071500	0.60624740
0.40	0.50	0.20	0.15	0.19301374	0.20751188	0.12929341	0.31039234	0.31306744	0.21106344
0.40	0.50	0.20	0.20	0.04937284	0.05633475	0.02693083	0.10832559	0.11047370	0.05927709
0.40	0.50	0.20	0.25	0.19301374	0.20751188	0.11412566	0.31039234	0.31306744	0.19335568
0.40	0.50	0.20	0.30	0.61608647	0.63402403	0.45253126	0.74880321	0.75071500	0.58393269
0.40	0.50	0.20	0.35	0.95319022	0.95740125	0.86568313	0.98051294	0.98079899	0.91929228
0.40	0.50	0.20	0.40	0.99999576	0.99999633	0.99997189	0.99999929	0.99999934	0.99999150
0.50	0.50	0.25	0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.50	0.50	0.25	0.05	0.99944150	0.99946009	0.97993910	0.99987127	0.99986701	0.98899443
0.50	0.50	0.25	0.10	0.93747953	0.94124227	0.83335268	0.97275669	0.97276226	0.89758518
0.50	0.50	0.25	0.15	0.59272404	0.60902149	0.45478093	0.73074541	0.73121329	0.58787505
0.50	0.50	0.25	0.20	0.18460856	0.19881200	0.12610682	0.30225040	0.30332959	0.21067523
0.50	0.50	0.25	0.25	0.04772485	0.05496134	0.02743712	0.10736593	0.10848780	0.06214233
0.50	0.50	0.25	0.30	0.18460856	0.19881200	0.10763999	0.30225040	0.30332959	0.19051813
0.50	0.50	0.25	0.35	0.59272404	0.60902149	0.41925915	0.73074541	0.73121329	0.56265654
0.50	0.50	0.25	0.40	0.93747953	0.94124227	0.81811214	0.97275669	0.97276226	0.89126217
0.50	0.50	0.25	0.45	0.99944150	0.99946009	0.97958631	0.99987127	0.99986701	0.98892929
0.50	0.50	0.25	0.50	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000

One escape from this dilemma can be seen by observing the following. Given β, $0 < \beta < 1$, and k, $0 < k < 1$, consider the value for n such that the probability of discovering a difference of k or more between p_{ij} and $p_{i.}p_{.j}$ is at least β. This value for n depends upon the values of $p_{1.}$ and $p_{.1}$. But the maximum value of n occurs for $p_{1.} = p_{.1} = 0.5$. So if we wish to choose n so that we will be at least 100β % certain of discovering a difference of at least k, we can choose n for the case $p_{1.} = p_{.1} = 0.5$.

There is also the problem of "being on the wrong side" of a non-symmetrical curve. However this is only a problem for very extreme (e.g. $p_{1.} = 0.8$, $p_{.1} = 0.2$) marginal probabilities as can be seen from the graphs in Appendix B.

Additional graphs and tables are given in Appendix B.

A great deal of symmetry can be observed in the graphs of the power curves. Any curve for which one of the marginal probabilities is 0.5 is symmetric about $p_{11} = p_{1.}p_{.1}$. Also, as might be expected, curves for $p_{1.} = a$, $p_{.1} = b$ and $p_{1.} = b$, $p_{.1} = a$ are identical; and curves for $p_{1.} = a$, $p_{.1} = b$ and $p_{1.} = a$, $p_{.1} = 1 - b$ are reflections. This means, for example, that to find the power from Table 1 when $p_{1.} = 0.6$ and $p_{.1} = 0.4$, the power is found from the $p_{1.} = 0.4$, $p_{.1} = 0.4$ entries.

EXAMPLE. Four hundred laboratory mice are inoculated for typhoid and placed in a large cage with 600 non-inoculated mice. All the mice are then exposed to typhoid.

A random sample is to be drawn from the population of mice and classified by using the table

	diseased	not diseased	total
inoculated			
not inoculated			
total			

The independence hypothesis is to be tested using the χ^2 statistic test with $\alpha = 0.05$. We wish to be at least 95% certain of discovering a high degree of association between row and column classifications.

Since the column probabilities are unknown, we use $p_{\cdot 1} = 0.5$. Then from Figure 3 we see that we can be 95% certain of discovering a 0.15 difference in p_{11} and $p_{1\cdot}p_{\cdot 1}$ with $n = 30$, a 0.12 difference with $n = 50$ and a 0.08 difference with $n \doteq 100$.

In a 2 x 2 table the constraint imposed by fixed marginal probabilities upon p_{11} is that p_{11} may range from max $\{0, p_{\cdot 1} - p_{2\cdot}\}$ to min $\{p_{1\cdot}, p_{\cdot 1}\}$. In this case the constraint is that p_{11} must satisfy the inequality $0.1 \leq p_{11} \leq 0.5$, as is illustrated by the graphs in Figure 3.

For this example the maximum possible difference between p_{ij} and $p_{i\cdot}p_{\cdot j}$ is 0.2, so that a sample of at least 100 should be drawn. Note that even with a sample of size 100, the probability is only 0.54 that a difference of 0.05, one fourth of the possible difference, will be discovered.

It should also be observed that the sampling must be done with replacement.

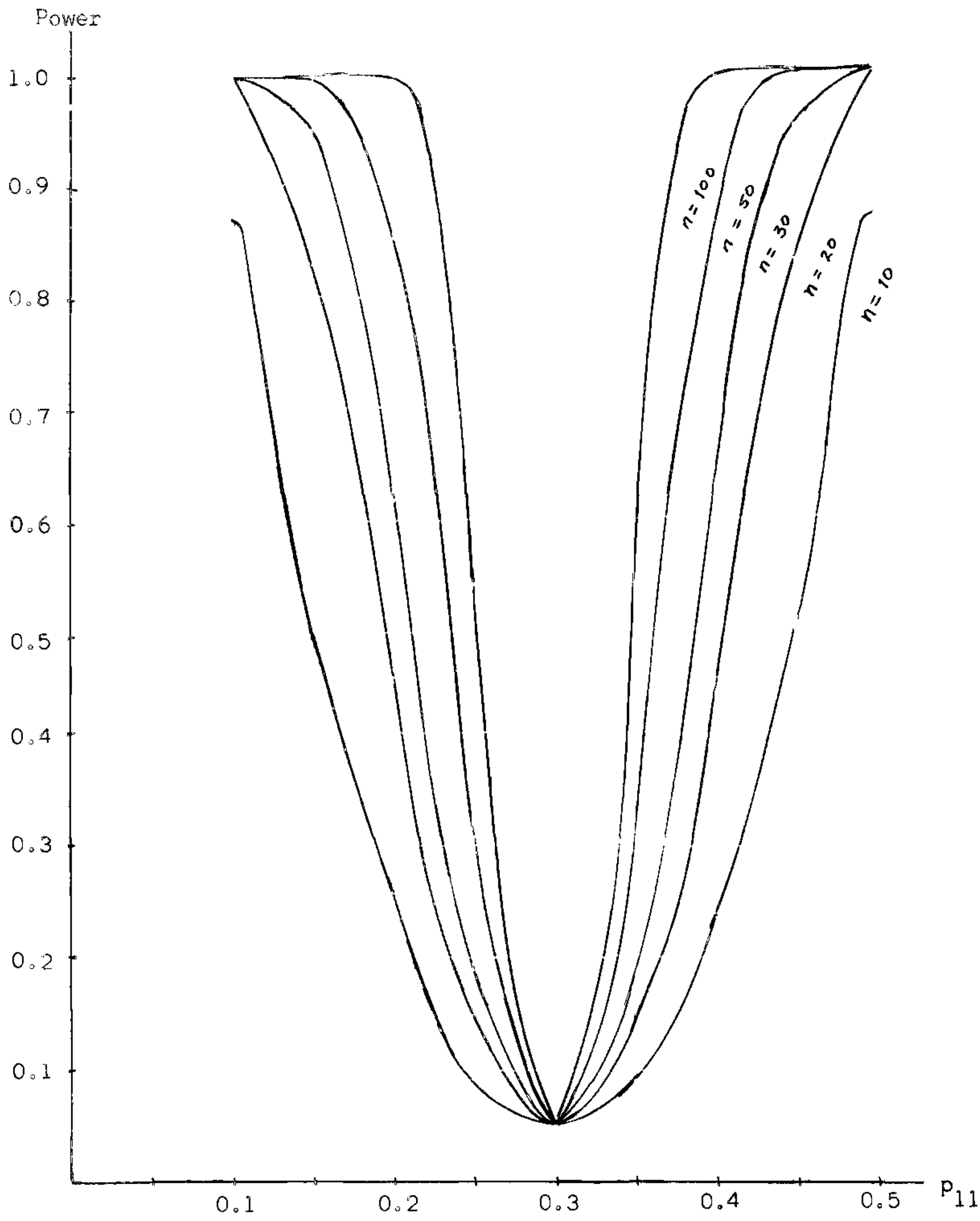

Figure 3. Power Curves for the χ^2 Statistic Test with $\alpha = 0.05$, $p_{1.} = 0.5$ and $p_{.1} = 0.6$.

CHAPTER V

EXTENSIONS TO HIGHER DIMENSIONS

Extending the results from Chapters II and III to higher dimensional contingency tables can become quite complex, and we shall not attempt to carry out all these extensions. Instead, we shall be content to present two or three representative cases.

The notation used previously can be modified in the natural way to apply to an $r \times s \times t$ contingency table. Thus n_{ijk} denotes the number from a total of n observations which fall into cell (i,j,k) of the $r \times s \times t$ table. Similarly, p_{ijk} denotes the multinomial probability that a single random observation falls into cell (i,j,k), and we have

$$\sum_{i,j,k} n_{ijk} = n$$

$$\sum_{i,j,k} p_{ijk} = 1 \ .$$

Marginal totals and probabilities are denoted

$$\sum_{i,j} n_{ijk} = n_{\cdot\cdot k}$$

$$\sum_{i,k} p_{ijk} = p_{\cdot j \cdot}$$

$$\sum_{k} n_{ijk} = n_{ij\cdot} \quad ,$$

for example.

We now investigate the statistic $-2 \log \lambda$ to be used in testing the hypothesis

$$p_{ijk} = p_{i..}p_{.j.}p_{..k}$$

that the three classifications are independent.

Denoting by T the event that the table with cell frequencies n_{ijk} occurs,

$$P(T) = \binom{n}{n_{111},\ldots,n_{rst}} \Pi\, p_{ijk}^{n_{ijk}} ,$$

and $\sup_{\Omega} P(T)$ occurs when $p_{ijk} = \hat{p}_{ijk} = \frac{n_{ijk}}{n}$. Under the independence hypothesis,

$$P(T) = \binom{n}{n_{111},\ldots,n_{rst}} \Pi\, p_{i..}^{n_{i..}}\ \Pi\, p_{.j.}^{n_{.j.}}\ \Pi\, p_{..k}^{n_{..k}}$$

and so $\sup_{\omega} P(T)$ occurs when $p_{i..} = \hat{p}_{i..} = \frac{n_{i..}}{n}$, $p_{.j.} = \hat{p}_{.j.} = \frac{n_{.j.}}{n}$ and $p_{..k} = \hat{p}_{..k} = \frac{n_{..k}}{n}$. Hence

$$\lambda = \frac{\Pi\, n_{i..}^{n_{i..}}\ \Pi\, n_{.j.}^{n_{.j.}}\ \Pi\, n_{..k}^{n_{..k}}}{n^{2n}\ \Pi\, n_{ijk}^{n_{ijk}}} .$$

The results from Chapter III extend directly, so that the statistic $-2 \log \lambda$ is asymptotically distributed as a chi-square random variable with

$$rst - 1 - (r + s + t - 3) = rst - r - s - t + 2$$

degrees of freedom.

Similarly,

$$\chi^2 = n^2 \sum \frac{\left(n_{ijk} - \frac{n_{i..}n_{.j.}n_{..k}}{n^2}\right)^2}{n_{i..}n_{.j.}n_{..k}}$$

is asymptotically distributed as a chi-square random variable with $rst - r - s - t + 2$ degrees of freedom.

There are other independence hypotheses of interest for an $r \times s \times t$ table. For example, the hypothesis that the "i" classification is independent of the "j" and "k" classification is stated

$$p_{ijk} = p_{i..}\, p_{.jk} .$$

The likelihood ratio is

$$\lambda = \frac{\Pi n_{i..}^{n_{i..}} \quad \Pi n_{.jk}^{n_{.jk}}}{n^n \quad \Pi n_{ijk}^{n_{ijk}}} ,$$

and $-2 \log \lambda$ is asymptotically distributed as a chi-square random variable with

$$rst - 1 - (r + st - 2) = (r - 1)(st - 1)$$

degrees of freedom.

For further discussion of the chi-square test for multi-dimensional contingency tables, see KASTENBAUM AND LAMPHIEAR and NORTON. The exact test for three-dimensional tables is discussed in FREEMAN AND HALTON.

A rather general discussion of interactions in multidimensional contingency tables is found in GOODMAN.

APPENDIX A

THE PEARSON-FISHER CHI-SQUARE THEOREM

This theorem was first formulated correctly by Sir Ronald A. Fisher (FISHER [1]), thereby setting off his enduring quarrel with Karl Pearson, whose earlier version of the theorem required correction. The first substantial analytic proof was given by CRAMÉR.

The formulation given here is by BIRCH, and the proofs of the first six lemmas are indicated by Birch. This approach differs from Cramér's principally in the criterion for the maximum likelihood estimate and in the differentiability requirement. The Birch formulation is the more general.

A function and its argument will be vector-valued with real components. It will be clear from the context when more specialized functions are being considered. The arguments of a function will be omitted at times, when it is not confusing to do so.

The Pearson-Fisher Theorem

Hypotheses:

H1. $\pi(\theta) = (\pi_1(\theta), \ldots, \pi_r(\theta))$ is defined for $\theta \in \Theta$ where $\Theta \subset E_s$.

H2. $\pi_i(\theta) \geq 0$ for $i = 1, \ldots, r$ and $\theta \in \Theta$.

H3. $\Sigma\, \pi_i(\theta) = 1$ for $\theta \in \Theta$.

H4. θ_o is an interior point of Θ.

H5. Given $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $|\theta - \theta_o| > \varepsilon$, $|\pi(\theta) - \pi(\theta_o)| > \delta$.

H6. $\pi_i(\theta_o) > 0$ for $i = 1,\ldots,r$.

H7. For each i numbers a_{ij} exist such that

$$\pi_i(\theta) = \pi_i(\theta_o) + \sqrt{\pi_i(\theta_o)} \sum_j a_{ij}(\theta_j - \theta_{oj}) + o(|\theta - \theta_o|)$$

as $\theta \to \theta_o$, i.e. each $\pi_i(\theta)$ is totally differentiable at θ_o with partial derivatives

$$\frac{\partial}{\partial\theta_j} \pi_i(\theta_o) = a_{ij} \sqrt{\pi_i(\theta_o)} .$$

H8. The matrix $A = [a_{ij}]$ has rank s, or equivalently, the matrix of partials has rank s.

H9. $\{X_k\}$, $k = 1,2,\ldots$ is a sequence of independent random variables, each taking the value i with probability $\pi_{oi} = \pi_i(\theta_o)$. P_{ni} is the proportion of X's in the first n trials taking the value i.

H10. For each n, there is a sequence $\{\theta_{nm}\}$, $m = 1,2,\ldots,$ with the property that $\theta_{nm} \in \Theta$ for each m, and there is a $\hat{\theta}_n \in \Theta$ such that $\theta_{nm} \to \hat{\theta}_n$ and

$$2n \sum_i [P_{ni} \log \pi_i(\theta_{nm}) - P_{ni} \log P_{ni}] \to \sup_{\theta \in \Theta} 2n \sum_i [P_{ni} \log \pi_i(\theta) - P_{ni} \log P_{ni}]$$

as $m \to \infty$.

Conclusion:

As $n \to \infty$, the random variables $\sqrt{n}\,(\hat{\theta}_{n1} - \theta_{o1})$, $\sqrt{n}\,(\hat{\theta}_{n2} - \theta_{o2}),\ldots,$

$\sqrt{n}(\hat{\theta}_{ns} - \theta_{os})$ and

$$n \sum \frac{[P_{ni} - \pi_i(\hat{\theta}_n)]^2}{\pi_i(\hat{\theta}_n)}$$

tend in distribution to random variables $Z_1,\ldots,Z_s$ and Y, where $(Z_1,\ldots,Z_s)$ is $N(0,(A^TA)^{-1})$ and Y is χ^2 with $r-s-1$ degrees of freedom. Moreover, the set $(Z_1,\ldots,Z_n)$ is independent of Y. That is, $\sqrt{n}(\hat{\theta}_n - \theta_o)$ is asymptotically normally distributed with zero mean vector and covariance matrix $(A^TA)^{-1}$;

$$n \sum \frac{[P_{ni} - \pi_i(\hat{\theta}_n)]^2}{\pi_i(\hat{\theta}_n)}$$

is asymptotically χ^2 with $r-s-1$ degrees of freedom; and $\sqrt{n}(\hat{\theta}_n - \theta_o)$ and

$$n \sum \frac{[P_{ni} - \pi_i(\hat{\theta}_n)]^2}{\pi_i(\hat{\theta}_n)}$$

are asymptotically independent.

Some remarks are perhaps in order before proceeding to the proof of this theorem.

(1) The probability of the outcome $P_{n1},\ldots,P_{nr}$ in the first n trials, where the probability of outcome i on each trial is π_i, is

$$P = \binom{n}{nP_{n1},\ldots,nP_{nr}} \prod_{i=1}^{r} \pi_i^{nP_{ni}} .$$

Then $\log P = \log n! + n \Sigma P_{ni} \log \pi_i - \Sigma \log (nP_{ni})!$. So if π_i is a function of θ, P is a maximum when $\Sigma P_{ni} \log \pi_i(\theta)$ is maximum, which will occur at any value of θ for which

$$2n \sum [P_{ni} \log \pi_i(\theta) - P_{ni} \log P_{ni}]$$

is a maximum. Therefore, the latter may be construed as a likelihood function.

(2) $n \sum \frac{[P_{ni} - \pi_i(\hat{\theta}_n)]^2}{\pi_i(\hat{\theta}_n)}$ is the $\sum \frac{(O - E)^2}{E}$ statistic, the so-called chi-square goodness-of-fit statistic.

(3) H5 is equivalent to saying that π^{-1} is continuous at θ_o.

(4) $A^T A$ is Fisher's information matrix for θ at θ_o.

(5) H7 can be replaced by the stronger but ordinarily more easily verified condition H7(a) that $\frac{\partial \pi_i}{\partial \theta_j}$ is continuous at θ_o for each i and j.

(6) $\bar{\Theta}$ denotes the closure of Θ in E_s if Θ is bounded and the closure of Θ in E_s plus a point at ∞ if Θ is not bounded. So $\hat{\theta}_n$ may be infinite if Θ is not bounded.

(7) If the maximum likelihood estimate is defined as it is here (in H10), there is always at least one such estimate, though it may be a point at which π has not been defined. There may be several points or a whole interval, but in practice, it is usually found that there is only one.

Proof of the Theorem

We first prove a series of lemmas. The subscript i will assume the values $1,\ldots,r$ and the subscript j will assume the values $1,\ldots,s$ throughout.

Lemma 1. Let $p_1,\ldots,p_r$ be any non-negative numbers such that $\Sigma p_i = 1$. Then

$$-2\sum [p_i \log \pi_i(\theta) - p_i \log p_i] \geq |p - \pi(\theta)|^2 .$$

Proof: Let $f(x) = x \log x$. Then $f'(x) = 1 + \log x$ and $f''(x) = \frac{1}{x}$, and by Taylor's theorem, for p_i and $\pi_i > 0$,

$$p_i \log p_i = \pi_i \log \pi_i + (1 + \log \pi_i)(p_i - \pi_i) + \frac{1}{2w_i}(p_i - \pi_i)^2$$

where w_i is between p_i and π_i. So

$$-2 [p_i \log \pi_i - p_i \log p_i] - 2(p_i - \pi_i) = \frac{1}{w_i}(p_i - \pi_i)^2 \geq (p_i - \pi_i)^2$$

since w_i between p_i and π_i implies that $0 < w_i < 1$. This inequality also holds for p_i and π_i non-negative if we interpret $a \log b$ as 0 if $a = o$ and $b \geq o$ and as $-\infty$ if $b = o$ and $a > o$. For then,

$$-2[p_i \log \pi_i - p_i \log p_i] - 2(p_i - \pi_i) \geq (p_i - \pi_i)^2$$

if $p_i = 0$ or $\pi_i = 0$. Now $\Sigma p_i = \Sigma \pi_i = 1$, so

$$-2\,\Sigma[p_i \log \pi_i - p_i \log p_i] \geq \Sigma(p_i - \pi_i)^2 \geq |p - \pi|^2 ,$$

which completes the proof of Lemma 1. This property is essentially a convexity property and could be proved using Jensen's inequality.

Lemma 2. Let $p_1,\ldots,p_r$ be as in Lemma 1. Let

$$y_i = \frac{(p_i - \pi_{oi})}{\sqrt{\pi_{oi}}} \quad \text{for} \quad i = 1,\ldots,r .$$

Then, as $p \rightarrow \pi_o$ and $\theta \rightarrow \theta_o$

$$-2\sum [p_i \log \pi_i(\theta) - p_i \log p_i] = [y - A(\theta - \theta_o)]^T [y - A(\theta - \theta_o)]$$

$$+ o(|y|^2 + |\theta - \theta_o|^2) .$$

Proof: Since $\pi_{oi} > 0$ (H6) and $\pi_i(\theta)$ is continuous at θ_o (H7), it is sufficient to restrict attention to positive p_i and π_i. Now, as in Lemma 1,

$$-2\sum [p_i \log \pi_i - p_i \log p_i] = \sum \frac{1}{w_i} [\pi_i - p_i]^2 ,$$

where w_i is between p_i and π_i. By virtue of the continuity of $\frac{1}{x}$ at positive x it follows that

$$\frac{1}{w_i} = \frac{1}{\pi_i} + o(1) ,$$

where $\lim_{p_i \to \pi_i} o(1) = 0$; and by virtue of the continuity of $\pi_i(\theta)$ at θ_o,

$$\frac{1}{\pi_i} = \frac{1}{\pi_{oi}} + o(1) ,$$

where $\lim_{\theta \to \theta_o} o(1) = 0$. Thus

$$\frac{1}{w_i} = \frac{1}{\pi_{oi}} + o(1) \quad \text{as} \quad \theta \to \theta_o \quad \text{and} \quad p \to \pi_o .$$

Also, since $(a + b)^2 \leq 2a^2 + 2b^2$ for real a and b,

$$(\pi_i - p_i)^2 \leq 2(\pi_i - \pi_{oi})^2 + 2(p_i - \pi_{oi})^2$$

for each i.

From H7, $\pi_i - \pi_{oi} = \sqrt{\pi_i(\theta_{oi})} \sum_j a_{ij}(\theta_j - \theta_{oj}) + o(|\theta - \theta_o|)$ so that

$$\frac{|\pi_i - \pi_{oi}|}{|\theta - \theta_o|} \leq \sqrt{\pi_i(\theta_o)} \sum |a_{ij}| \frac{|\theta_j - \theta_{oj}|}{|\theta - \theta_o|} + o(1)$$

$$\leq \sqrt{\pi_i(\theta_o)} \sum |a_{ij}| + o(1) \ .$$

Thus $\pi_i - \pi_{oi}$ is $O(|\theta - \theta_o|)$, i.e. the ratio of $\pi_i - \pi_{oi}$ to $|\theta - \theta_o|$ becomes and remains bounded as $\theta \to \theta_o$.

From the definition of y $p_i - \pi_{oi}$ is $O(|y|)$. Hence $(\pi_i - p_i)^2$ is $O(|\theta - \theta_o|^2 + |y|^2)$ for each i and

(1) $$\sum \frac{1}{w_i} (\pi_i - p_i)^2 = \sum_i \left(\frac{1}{\pi_{oi}} + o(1)\right)(\pi_i - p_i)^2$$

$$= \sum_i \frac{1}{\pi_{oi}} (\pi_i - p_i)^2 + o(|\theta - \theta_o|^2 + |y|^2)$$

Subtracting $\sqrt{\pi_{oi}}\ y_i$ from both sides of the equation in H7 we have

$$\pi_i - p_i = \sqrt{\pi_{oi}} \left[\sum_j [a_{ij}(\theta_j - \theta_{oj})] - y_i\right] + o(|\theta - \theta_o|) \ .$$

and so

$$\sum_i \frac{1}{\pi_{oi}} (\pi_i - p_i)^2 = \sum_i \left[y_i - \sum_j a_{ij}(\theta_j - \theta_{oj}) + o(|\theta - \theta_o|)\right]^2$$

$$= \sum_i \left[y_i - \sum_j a_{ij}(\theta_j - \theta_{oj})\right]^2 + o[(|y| + |\theta - \theta_o|)|\theta - \theta_o|]$$

$$(2) \qquad = \sum_i \left[y_i - \sum_j a_{ij}(\theta_j - \theta_{oj}) \right]^2 + o[|y|^2 + |\theta - \theta_o|^2] \; .$$

The second line above follows from these facts:

$$y_i - \sum_j a_{ij}(\theta_j - \theta_{oj}) \quad \text{is} \quad O(|y| + |\theta - \theta_o|) \; ;$$

$$O(|y| + |\theta - \theta_o|)\, o(|\theta - \theta_o|) \quad \text{is} \quad o[(|y| + |\theta - \theta_o|)|\theta - \theta_o|] \; ;$$

and

$$[|y| + |\theta - \theta_o|]|\theta - \theta_o| \geq |\theta - \theta_o|^2 \; .$$

To get (2), we observe that

$$[|y| + |\theta - \theta_o|]|\theta - \theta_o| = \frac{1}{2}|y|^2 + \frac{3}{2}|\theta - \theta_o|^2 - \frac{1}{2}(|y| - |\theta - \theta_o|)^2$$

$$\leq \frac{3}{2}(|y|^2 + |\theta - \theta_o|^2) \; .$$

Finally,

$$(3) \qquad \sum_i [y_i - \sum_j a_{ij}(\theta_j - \theta_{oj})]^2 = [y - A(\theta - \theta_o)]^T [y - A(\theta - \theta_o)]$$

and the lemma follows from (1), (2) and (3).

<u>Lemma 3.</u> Let $\theta^*(p) = (A^T A)^{-1} A^T y + \theta_o$. Then

$$-2 \sum [p_i \log \pi_i(\theta) - p_i \log p_i]$$

$$= R + (\theta - \theta^*(p))^T A^T A(\theta - \theta^*(p)) + o(|y|^2 + |\theta - \theta^*(p)|^2)$$

as $p \to \pi_o$ and $\theta \to \theta_o$, where

$$R = [y - A(\theta^*(p) - \theta_o)]^T [y - A(\theta^*(p) - \theta_o)]$$

$$= y^T y - y^T A (A^T A)^{-1} A^T y .$$

Proof. We note that $(A^T A)^{-1}$ exists because of H8. Now

$$[y - A(\theta - \theta_o)]^T [y - A(\theta - \theta_o)] = y^T y - y^T A(\theta - \theta_o) - (\theta - \theta_o)^T A^T y +$$

$$+ (\theta - \theta_o)^T A^T A(\theta - \theta_o)$$

and

$$(\theta - \theta^*)^T A^T A(\theta - \theta^*) = [\theta - (A^T A)^{-1} A^T y - \theta_o]^T A^T A[\theta - (A^T A)^{-1} A^T y - \theta_o]$$

$$= (\theta - \theta_o)^T A^T A(\theta - \theta_o) - (\theta - \theta_o)^T A^T A(A^T A)^{-1} A^T y - y^T A[(A^T A)^{-1}]^T A^T A(\theta - \theta_o) +$$

$$+ y^T A[(A^T A)^{-1}]^T A^T A(A^T A)^{-1} A^T y$$

$$= (\theta - \theta_o)^T A^T A(\theta - \theta_o) - (\theta - \theta_o)^T A^T y - y^T A(\theta - \theta_o) + y^T A(A^T A)^{-1} A^T y$$

Therefore,

$$[y - A(\theta - \theta_o)]^T [y - A(\theta - \theta_o)] = R + (\theta - \theta^*)^T A^T A(\theta - \theta^*)$$

and from Lemma 2, it follows that

$$- 2 \sum_i [p_i \log \pi_i - p_i \log p_i] = R + (\theta - \theta^*)^T A^T A(\theta - \theta^*) + o(|y|^2 + |\theta - \theta_o|^2).$$

But $|y|^2 + |\theta - \theta_o|^2 \leq 2[|y|^2 + |\theta - \theta^*|^2 + |\theta^* - \theta_o|^2]$ and $\theta^* - \theta_o$ is $O(|y|)$. The desired result follows.

Lemma 4. Let $\hat{\theta}(p)$ be any value of θ for which there exists a sequence $\{\theta_m(p)\}$ such that $\theta_m(p) \to \hat{\theta}(p)$ and

$$n \sum [p_i \log \pi_i(\theta_m(p)) - p_i \log p_i] \to \sup_{\theta \in \Theta} n \sum [p_i \log \pi_i(\theta) - p_i \log p_i] .$$

Then

$$\hat{\theta}(p) - \theta^*(p) = o(|y|) \quad \text{as} \quad p \to \pi_o ,$$

i.e. $\hat{\theta}(p) = \theta_o + (A^T A)^{-1} A^T y + o(|y|)$ as $p \to \pi_o$.

Proof: We first note that

$$\sup_{\theta \in \Theta} \sum [p_i \log \pi_i(\theta) - p_i \log p_i] = - \inf_{\theta \in \Theta} - \sum [p_i \log \pi_i(\theta) - p_i \log p_i] .$$

Also, we note that $\inf_{\theta \in A} f(\theta) \geq \inf f(\theta)$, and that $\inf_{\theta \in A} f(\theta) > \inf f(\theta)$ implies that $\inf_{\theta \in A^c} f(\theta) = \inf f(\theta)$, where A^c is the complement of A.

Then it is sufficient to prove that, given $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $|p - \pi_o| < \delta$,

$$\inf_{|\theta - \theta^*| > \varepsilon |y|} - 2 \sum [p_i \log \pi_i(\theta) - p_i \log p_i] > -2 \sum [p_i \log \pi_i(\theta^*) - p_i \log p_i],$$

since

$$-2 \sum [p_i \log \pi_i(\theta^*) - p_i \log p_i] \geq \inf_{\theta \in \Theta} -2 \sum [p_i \log \pi_i(\theta) - p_i \log p_i] .$$

For then, $|\hat{\theta} - \theta^*| \leq \varepsilon|y|$ whenever $|p - \pi_o| < \delta$ since

$$-2\sum[p_i \log \pi_i(\theta) - p_i \log p_i]$$

tends to its inf as θ tends to $\hat{\theta}$.

Now A^TA is non-singular by H8. Furthermore, A^TA is positive definite. In fact, $x^TA^TAx = y^Ty \geq 0$, where $y = Ax$. Now $y^Ty = 0$ if and only if $y = 0$, and since y is a linear combination of the linearly independent columns of A, $y = 0$ if and only if $x = 0$. Let λ be the smallest eigenvalue of A^TA. Then $\lambda > 0$.

Choose $\delta_1 > 0$ and $\eta > 0$ such that, whenever $|p - \pi_o| < \delta_1$ and $|\theta - \theta^*(p)| < \eta$ and $\theta \in \Theta$, then (see Lemma 3)

$$(4) \qquad R + (\theta - \theta^*)^T A^T A(\theta - \theta^*) - \frac{\varepsilon^2\lambda}{\varepsilon^2 + 2}(|y|^2 + |\theta - \theta^*|^2)$$

$$< -2\sum[p_i \log \pi_i(\theta) - p_i \log p_i]$$

$$< R + (\theta - \theta^*)^T A^T A(\theta - \theta^*) + \frac{\varepsilon^2\lambda}{\varepsilon^2 + 2}(|y|^2 + |\theta - \theta^*|^2) .$$

It is possible to so choose δ_1 and η because from Lemma 3

$$|\theta - \theta_o|^2 \leq 2[|\theta - \theta^*|^2 + O(|y|^2)]$$

and, as $p \to \pi_o$, $y \to 0$.

So then, for $\varepsilon|y| < |\theta - \theta^*| < \eta$ and $|p - \pi_o| < \delta_1$,

$$- 2 \sum [p_i \log \pi_i(\theta) - p_i \log p_i] > R + \lambda |\theta - \theta^*|^2 - \frac{\varepsilon^2 \lambda}{\varepsilon^2 + 2} (|y|^2 + |\theta - \theta^*|^2)$$

which in turn is greater than

$$R + \left(\lambda - \frac{\varepsilon^2 \lambda}{\varepsilon^2 + 2}\right) \varepsilon^2 |y|^2 - \frac{\varepsilon^2 \lambda}{\varepsilon^2 + 2} |y|^2 = R + \frac{\varepsilon^2 \lambda}{\varepsilon^2 + 2} |y|^2 ,$$

while

$$- 2 \sum_i [p_i \log \pi_i(\theta^*) - p_i \log p_i] < R + \frac{\varepsilon^2 \lambda}{\varepsilon^2 + 2} |y|^2 ,$$

by direct substitution of θ^* for θ in (4).

Therefore, when $|p - \pi_o| < \delta_1$,

$$\inf_{\eta > |\theta - \theta^*| > \varepsilon |y|} - 2 \sum [p_i \log \pi_i(\theta) - p_i \log p_i]$$

$$> - 2 \sum [p_i \log \pi_i(\theta^*) - p_i \log p_i] .$$

Because of H5 there exists an $\eta' > 0$ such that $|\pi - \pi_o| > \eta'$ whenever $|\theta - \theta_o| > \frac{1}{2} \eta$. Now as $p \to \pi_o$, $y \to 0$, and therefore $\theta^* \to \theta_o$, and Lemma 3 yields

$$- 2 \sum [p_i \log \pi_i(\theta^*) - p_i \log p_i] \to 0 \text{ as } p \to \pi_o .$$

We may then choose a number $0 < \delta_2 < \frac{1}{2} \eta'$ such that $|\theta^* - \theta_o| < \frac{1}{2} \eta$ and

$$- 2 \sum [p_i \log \pi_i(\theta^*) - p_i \log p_i] < \frac{1}{4} \eta'^2$$

whenever $|p - \pi_o| < \delta_2$. Now, when $|\theta - \theta^*| \geq \eta$ and $|p - \pi_o| < \delta_2$, $|\theta - \theta_o| \geq |\theta - \theta^*| - |\theta^* - \theta_o| > \frac{1}{2}\eta$ and therefore $|\pi - \pi_o| > \eta'$.

From Lemma 1

$$- 2\sum[p_i \log \pi_i(\theta) - p_i \log p_i] \geq |p - \pi|^2 \geq [\,|\pi - \pi_o| - |p - \pi_o|\,]^2$$

$$> (\eta' - \delta_2)^2 > \frac{1}{4}\eta'^2 .$$

Therefore, when $|p - \pi_o| < \delta_2$,

$$\inf_{|\theta - \theta^*| \geq \eta} - 2\sum[p_i \log \pi_i(\theta) - p_i \log p_i] \geq - 2\sum[p_i \log \pi_i(\theta^*) - p_i \log p_i].$$

Finally, take $\delta = \min\{\delta_1, \delta_2\}$. For $|p - \pi_o| < \delta$,

$$\inf_{|\theta - \theta^*| > \varepsilon|y|} - 2\sum[p_i \log \pi_i(\theta) - p_i \log p_i]$$

$$> - 2\sum[p_i \log \pi_i(\theta^*) - p_i \log p_i],$$

which completes the proof of Lemma 4.

<u>Lemma 5.</u>

$$\sum \frac{1}{\pi_i(\theta)} (p_i - \pi_i(\theta))^2 = [\,y - A(\theta - \theta_o)\,]^T[\,y - A(\theta - \theta_o)]$$

$$+ o(|y|^2 + |\theta - \theta_o|^2)$$

as $\theta \to \theta_o$ and $p \to \pi_o$.

Proof: From Lemma 2,

$$\sum \frac{1}{\pi_i(\theta)} (p_i - \pi_i(\theta))^2 = \sum_i \frac{\pi_{oi}}{\pi_i(\theta)} [y_i - \sum_j a_{ij}(\theta_j - \theta_{oj}) + o(|\theta - \theta_o|)]^2.$$

The conclusion of the lemma follows from the proof of Lemma 2 after observing that from H7

$$\pi_i(\theta) = \pi_{oi} + O(|\theta_j - \theta_{oj}|),$$

and so

$$\frac{\pi_{oi}}{\pi_i(\theta)} = 1 + \frac{O(|\theta_j - \theta_{oj}|)}{\pi_{oi} + O(|\theta_j - \theta_{oj}|)} = 1 + O(1)$$

as $\theta \to \theta_o$.

Lemma 6.

$$\sum_i \frac{1}{\pi_i(\hat\theta)} (p_i - \pi_i(\hat\theta))^2 = y^T y - y^T A(A^T A)^{-1} A^T y + o(|y|^2)$$

as $p \to \pi_o$.

Proof: From Lemma 5 and Lemma 3,

$$\sum \frac{1}{\pi_i(\hat\theta)} (p_i - \pi_i(\hat\theta))^2 = R + (\hat\theta - \theta^*) A^T A(\hat\theta - \theta^*) + o(|y|^2 + |\hat\theta - \theta_o|^2).$$

But $\hat\theta - \theta^* = o(|y|)$ as $p \to \pi_o$ from Lemma 4; and, since

$$|\hat\theta - \theta_o|^2 \leq 2[|\hat\theta - \theta^*|^2 + |\theta^* - \theta_o|^2]$$

and $\theta^* - \theta_o$ is $O(|y|)$, we have

$$\sum \frac{1}{\pi_i(\hat{\theta})} (p_i - \pi_i(\hat{\theta}))^2 = R + o(|y|^2) \quad \text{as} \quad p \to \pi_o ,$$

which is what was to be proved.

Lemma 7. Define Y_n by $Y_{ni} = \dfrac{(P_{ni} - \pi_{oi})}{\sqrt{\pi_{oi}}}$.

Then the joint distribution of $\sqrt{n}\ Y_{n1}, \ldots, \sqrt{n}\ Y_{nr}$ tends as $n \to \infty$ to a (singular) multivariate normal distribution with mean 0 and covariance matrix $I_r - vv^T$, where $v^T = (\sqrt{\pi_{o1}}, \ldots, \sqrt{\pi_{or}})$.

Proof: (Using the method of characteristic functions.) Recall the independence in H9.

Let $\xi_j = (\xi_{j1}, \ldots, \xi_{jr})$ for $j = 1, \ldots, n$, where

$$\xi_{jk} = \begin{cases} \dfrac{1 - \pi_{ok}}{\sqrt{n\ \pi_{ok}}} & \text{if } X_j \text{ has the value } k \\[2ex] \dfrac{-\pi_{ok}}{\sqrt{n\ \pi_{ok}}} & \text{otherwise .} \end{cases}$$

Then $\sum \xi_j = (\sqrt{n}\, Y_{n1}, \ldots, \sqrt{n}\ Y_{nr}) = Y_n$. The characteristic function for ξ_j is

$$\varphi_j(t) = E(e^{it^T\xi_j}) = \pi_{o1} e^{i\left(\frac{1-\pi_{oi}}{\sqrt{n\pi_{o1}}} t_1 - \frac{\pi_{o2}}{\sqrt{n\pi_{o2}}} t_2 - \cdots - \frac{\pi_{or}}{\sqrt{n\pi_{or}}} t_r\right)} + \cdots + \pi_{or} e^{i\left(\frac{-\pi_{o1}}{\sqrt{n\pi_{o1}}} t_1 - \cdots - \frac{1-\pi_{or}}{\sqrt{n\pi_{or}}} t_r\right)}$$

$$= \pi_{o1} e^{i \frac{t_1}{n\pi_{o1}}} e^{-i \Sigma \frac{\sqrt{\pi_{ok}}}{\sqrt{n}} t_k} + \ldots + \pi_{or} e^{i \frac{t_r}{\sqrt{n\pi_{or}}}} e^{-i \Sigma \frac{\sqrt{\pi_{ok}}}{\sqrt{n}} t_k}$$

$$= e^{-\frac{i}{\sqrt{n}} \Sigma \sqrt{\pi_{ok}}\, t_k} \left(\sum \pi_{ok} e^{i \frac{t_k}{\sqrt{n\pi_{ok}}}} \right),$$

which means that all the φ_j's are equal.

Now, since $Y_n = \sum \xi_j$ and the ξ_j's are independent, the characteristic function for Y_n is given by

$$\varphi(t) = \prod_j \varphi_j(t) = e^{-i\sqrt{n}\, \Sigma \sqrt{\pi_{ok}}\, t_k} \left(\sum \pi_{ok} e^{i \frac{t_k}{\sqrt{n\pi_{ok}}}} \right)^n,$$

since the characteristic function of a sum of independent random variables is the product of the characteristic functions of the independent random variables (see LOÈVE, p. 227).

We wish to investigate the behavior of φ as $n \to \infty$.

$$\log \varphi = n \log \left(\pi_{o1} e^{i \frac{t_1}{\sqrt{n\pi_{o1}}}} + \ldots + \pi_{or} e^{i \frac{t_r}{\sqrt{n\pi_{or}}}} \right) - i \sqrt{n} \sum t_k \sqrt{\pi_{ok}} .$$

Let $f(t) = \sum \pi_{ok}\, e^{i \frac{t_k}{\sqrt{n\pi_{ok}}}}$. The Maclaurin expansion gives

$$f(t) = 1 + \frac{i}{\sqrt{n}} \sum t_k \sqrt{\pi_{ok}} - \frac{1}{2n} \sum \pi_{ok}{}^2 + O(n^{-3/2}) .$$

So

$$\log \varphi = \frac{\log\left[1 + \frac{i}{\sqrt{n}} \sum t_k \sqrt{\pi_{ok}} - \frac{1}{2n} \sum t_k^2 + O(n^{-3/2})\right] - \frac{i}{\sqrt{n}} \sum t_k \sqrt{\pi_{ok}}}{\frac{1}{n}}$$

from which it follows that

$$\lim_{n \to \infty} \log \varphi = -\frac{1}{2} \sum t_k^2 + \frac{1}{2}\left(\sum t_k \sqrt{\pi_{ok}}\right)^2 .$$

To see this it should be observed that, if a function g has second derivatives in a neighborhood of zero and g'' is continuous at zero, then by Taylor's theorem

$$g(x) = g(0) + g'(0)x + g''(\xi)\frac{x^2}{2} \text{ where } \xi \text{ is between } 0$$

and x. Then

$$g(x) = g(0) + g'(0)x + g''(0)\frac{x^2}{2} + [g''(\xi) - g''(0)]\frac{x^2}{2}$$

$$= g(0) + g'(0)x + g''(0)\frac{x^2}{2} + o(x^2) ,$$

since $\lim_{x \to 0} [g''(\xi) - g''(0)] = 0$. Applying this to $\log (1 + x)$, we obtain

$$\log (1 + x) = x - \frac{x^2}{2} + o(x^2) ,$$

and letting $x = \frac{i}{\sqrt{n}} \sum t_k \sqrt{\pi_{ok}} - \frac{1}{2n} \sum t_k^2 + O(n^{-3/2})$ gives

$$\log \varphi = \left[x - \frac{x^2}{2} + o\left(\frac{1}{n}\right) - \frac{1}{\sqrt{n}} \sum t_k \sqrt{\pi_{ok}} \right] / \frac{1}{n}$$

$$= -\frac{1}{2} \sum t_k^{\,2} + \frac{1}{2}\left(\sum t_k \sqrt{\pi_{ok}}\right)^2 + \frac{o(\frac{1}{n})}{\frac{1}{n}}$$

Then, upon taking the limit as $n \to \infty$, the above result follows. Evidently, then

$$\lim_{n \to \infty} \varphi(t) = e^{-\frac{1}{2}\left[\Sigma t_k^{\,2} - (\Sigma t_k \sqrt{\pi_{ok}})^2\right]}$$

$$= e^{-\frac{1}{2}\left[t^T (I_r - v v^T) t\right]} .$$

The desired result follows from the Lévy continuity theorem for characteristic functions (see LOÈVE, p. 191), which implies that if a sequence of characteristic functions converges to the characteristic function of a random variable with distribution function F, then the corresponding sequence of random variables converges to a random variable with distribution function F.

Lemma 8. Let $Z_n^T = (Z_{n1}, \ldots, Z_{nk})$ and suppose that the joint distribution of the Z_{ni}'s tends (as $n \to \infty$) to a multivariate normal distribution with mean 0 and covariance matrix C, where C is $k \times k$ and not necessarily nonsingular. Let A^T be an $m \times k$ matrix and let $W_n = A^T Z_n$. Then the distribution of W_n tends to $N(0, A^T C A)$.

Proof: Let φ_{W_n} be the characteristic function for W_n. Then

$$\varphi_{W_n}(t) = E(e^{it^T W_n}) = E(e^{it^T A^T Z_n})$$

$$= E(e^{i(At)^T Z_n})$$

$$= E(e^{it'^T Z_n}) \quad \text{where} \quad t' = At \; .$$

But $E(e^{it'^T Z_n}) \to e^{-it'^T C t'}$. So $\varphi_{W_n}(t) \to e^{-it^T A^T C At}$, which is the characteristic function of $N(0, A^T CA)$ variables (since $A^T CA$ is non-negative definite if C is).

Lemma 9. (Helly-Bray Theorem). If g is complex-valued on E_k, $|g(x)|$ is bounded and $F_n \xrightarrow{c} F$, i.e. the distribution functions F_n converge to the distribution function F, then

$$\int_{E_k} g(x) \, dF_n(x) \to \int_{E_k} g(x) \, dF(x) \; .$$

Proof: See LOÉVE, p. 182.

Corollary. If $Z_n \xrightarrow{\text{dist}} N(0, I_k)$ then

$$\sum_{j=1}^{k} Z_{nj}^2 \xrightarrow{\text{dist}} \chi_k^2 \; .$$

Proof: Let $W_n = \sum_{j=1}^{k} Z_{nj}^2$.

$$\varphi_{W_n}(t) = E(e^{itW_n})$$

$$= \int_{E_k} e^{it \Sigma Z_{nj}^2} \, dF_n \to \int_{E_k} e^{it \Sigma Z_{nj}^2} \frac{e^{-\frac{1}{2} \Sigma Z_{nj}^2}}{(2\pi)^{k/2}} \, dZ$$

$$= \prod_{j=1}^{k} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx_j^2} e^{-\frac{x_j^2}{2}} \, dx_j$$

$$= \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}(1 - 2it)} dx \right]^k$$

$$= (1 - 2it)^{-k/2} = \varphi_{\chi_k^2}(t) .$$

Lemma 10. (Slutsky's Theorem). Let $X_n \xrightarrow{\text{dist}} X$ and $Y_n \to c$ in probability, c a constant. Then, if $Z_n = X_n + Y_n$,

$$Z_n \xrightarrow{\text{dist}} X + c, \quad \text{i.e.} \quad P(Z_n \leq t) \to P(X + c \leq t).$$

Proof: See CRAMÉR, p. 254.

Lemma 11. If $Z_n \xrightarrow{\text{dist}} N(0, I_k)$ and $Z_n^{*T} = (Z_{n1}, \ldots, Z_{nr})$ where $r < k$, then

$$Z_n^* \longrightarrow N(0, I_r).$$

Proof: Let $t^{*T} = (t_1, \ldots, t_r)$. Then

$$\varphi_{Z_n^*}(t^*) = E(e^{it^{*T}Z_n^*}) = E(e^{i[t^{*T} \,\vdots\, 0]Z_n})$$

$$= e^{-i[t^{*T} \,\vdots\, 0]\, I_k \left[\begin{smallmatrix} t^* \\ \cdots \\ 0 \end{smallmatrix}\right]}$$

$$= e^{-it^{*T}I_r t^*}$$

which is the characteristic function for (joint) $N(0, I_r)$ variables.

Lemma 12. If $W_n \xrightarrow{\text{dist}} N\left(0, \left[\begin{array}{c:c} I_{r-1} & 0 \\ \hdashline 0 & 0 \end{array}\right]\right)$ and $W_n^{*T} = (W_{n1}, \ldots, W_{n,r-1})$, then

$$W_n^* \xrightarrow{\text{dist}} N(0, I_{r-1}) .$$

Proof:

$$\varphi_{W_n^*}(t^*) = E(e^{it^{*T}W_n^*})$$

$$= E(e^{i[t^{*T} \,\vdots\, 0]W_n}) = \varphi_{W_n}(\begin{bmatrix} t^* \\ \cdots \\ 0 \end{bmatrix}) .$$

But $\varphi_{W_n}(t) \to e^{-\frac{1}{2} i\, t^T \begin{bmatrix} I_{r-1} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix} t}$

so $\varphi_{W_n^*}(t^*) \to e^{-\frac{1}{2} it^{*T} I_{r-1} t^*}$

which is the characteristic function for the $N(0, I_{r-1})$ distribution.

<u>Lemma 13</u>. Let $Z_n = \sqrt{n}\ Y_n$. Then

$$(A^TA)^{-1} A^T Z_n \xrightarrow{\text{dist}} N(0, (A^TA)^{-1})$$

and

$$Z_n^T Z_n - Z_n^T A(A^TA)^{-1}A^T Z_n \xrightarrow{\text{dist}} \chi^2_{r-s-1} .$$

Moreover, the random variables to which the two sequences converge are independent.

Proof. (A and v are the $r \times s$ matrix and $r \times 1$ vector used before.)

Let U be an $r \times r$ orthogonal matrix with last column v. Let $W_n = U^T Z_n$. (W_n is $r \times 1$.)

Then, from lemmas 7 and 8, $W_n \xrightarrow{\text{dist}} N(0, U^T(I_r - v v^T) U)$.

But

$$U^T(I_r - v v^T) U = U^T U - U^T v v^T U$$

$$= I_r - \left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c:c} I_{r-1} & 0 \\ \hdashline 0 & 0 \end{array}\right],$$

because the rows of U^T are each orthogonal to v, and $v \cdot v = 1$.

Hence

$$W_n \xrightarrow{\text{dist}} N\left(0, \left[\begin{array}{c:c} I_{r-1} & 0 \\ \hdashline 0 & 0 \end{array}\right]\right),$$

Now let $B = U^T A$. Note that

$$B^T B = A^T U U^T A = A^T A \; .$$

Since the rank of A is s, the rank of B is also s, and the s rows of B^T generate an s-dimensional vector space. Let $p^* = [p_1, \ldots, p_s]$ where p_i is $r \times 1$ and $\{p_1, \ldots, p_s\}$ is an orthonormal basis for this s-dimensional vector space. Then there is a non-singular $s \times s$ matrix α so that

$$B = p^* \alpha \; .$$

Choose $r \times 1$ vectors $p_{s+1},\ldots,p_r$ and let

$$p^{**} = [p_{s+1},\ldots,p_r]$$

so that

$$p = \begin{bmatrix} p^{*T} \\ \cdots \\ p^{**T} \end{bmatrix}$$

is an orthogonal $(r \times r)$ matrix. Now define

$$\eta_n^* = p^{*T} W_n \quad \text{and} \quad \eta_n^{**} = p^{**T} W_n \quad ,$$

and let

$$\eta_n = \begin{bmatrix} \eta_n^* \\ \cdots \\ \eta_n^{**} \end{bmatrix} = \begin{bmatrix} p^{*T} \\ \cdots \\ p^{**T} \end{bmatrix} W_n = p^T W_n .$$

Then

$$\eta_n \xrightarrow{\text{dist}} N\left(0,\ p^T \left[\begin{array}{c:c} I_{r-1} & 0 \\ \hdashline 0 & 0 \end{array}\right] p\right) = N\left(0,\ \left[\begin{array}{c:c} I_{r-1} & 0 \\ \hdashline 0 & 0 \end{array}\right]\right).$$

So $(\eta_{n1},\ldots,\eta_{n,r-1}) \xrightarrow{\text{dist}} N(0,\ I_{r-1})$ by Lemma 12 and

$$\eta_n^{*T} = (\eta_{n1},\ldots,\eta_{ns}) \longrightarrow N(0,\ I_s)$$

by Lemma 11.

Furthermore, $(\eta_{n,s+1},\ldots,\eta_{n,r-1}) \rightarrow N(0,\ I_{r-s-1})$ by Lemma 11 and so

$$\sum_{i=s+1}^{r-1} \eta_{n_i}^2 \xrightarrow{\text{dist}} \chi^2_{r-s-1}$$

by the corollary to Lemma 9.

But $\eta_{nr}^2 \to 0$ so

$$\eta_n^{**T}\ \eta_n^{**} = \sum_{i=s+1}^{r-1} \eta_{ni}^2 + \eta_{nr}^2 \xrightarrow{\text{dist}} \chi_{r-s-1}^2$$

by Lemma 10.

We now complete the proof by observing that

$$\begin{aligned} \eta_n^T \eta_n &= W_n^T pp^T W_n \\ &= Z_n^T U\, U^T Z_n \\ &= Z_n^T Z_n \end{aligned}$$

and also

$$\eta_n^T \eta_n = \eta_n^{*T}\ \eta_n^* + \eta_n^{**T}\ \eta_n^{**}$$

so that

$$Z_n^T Z_n - \eta_n^{*T} \eta_n^* = \eta_n^{**T}\ \eta_n^{**} .$$

Now

$$\begin{aligned} Z_n^T A(A^TA)^{-1}\ A^TZ_n &= W_n^T U^TUB(B^TB)^{-1}\ B^TU^TU\ W_n \\ &= W_n^T p^*\ \alpha(B^TB)^{-1}\ \alpha^T\ p^{*T}\ W_n \\ &= \eta_n^{*T} \alpha(\alpha^T\alpha)^{-1}\ \alpha^T\ \eta_n^* \\ &= \eta_n^{*T}\ \eta_n^* \end{aligned}$$

since $B^T B = \alpha^T p^{*T} p^* \alpha = \alpha^T \alpha$.

Hence

$$Z_n^T Z_n - Z_n^T A(A^T A)^{-1} A^T Z_n = \eta_n^{**T} \eta_n^{**}$$

and so

$$Z_n^T Z_n - Z_n^T A(A^T A)^{-1} A^T Z_n \xrightarrow{\text{dist}} \chi^2_{r-s-1} .$$

The limiting distribution for $(A^T A)^{-1} A^T Z_n$ follows from lemmas 7 and 8, and the fact that

$$(A^T A)^{-1} A^T [I_r - v v^T] A(A^T A)^{-1}$$

$$= (A^T A)^{-1} - (A^T A)^{-1} (A^T v)(v^T A)(A^T A)^{-1}$$

$$= (A^T A)^{-1}$$

since

$$A^T v = \left[\sum_{i=1}^{r} a_{ij} \sqrt{\pi_{oi}} \right]$$

$$= \left[\sum_{i=1}^{r} \frac{\partial \pi_{oi}}{\partial_j} \right]$$

$$= \left[\frac{\partial}{\partial_j} \sum_{i=1}^{r} \pi_{oi} \right]$$

$$= [\frac{\partial}{\partial_j} (1)]$$

$$= [\ 0\ ]_{s \times 1} .$$

To show independence, we have

$$(A^T A)^{-1} A^T Z_n = (B^T B)^{-1} B^T U^T U W_n$$

$$= (\alpha^T \alpha)^{-1} \alpha^T p^{*T} W_n$$

$$= \alpha^{-1} \eta_n^* .$$

Then $(A^T A)^{-1} A^T Z_n$ depends only upon η_n^* while $Z_n^T Z_n - Z_n^T A(A^T A)^{-1} A^T Z_n$ depends only upon η_n^{**}, and since η_n^* and η_n^{**} tend to variables which are independent, $(A^T A)^{-1} A^T Z_n$ and $Z_n^T Z_n - Z_n^T A(A^T A)^{-1} A^T Z_n$ tend to variables which are independent.

This completes the proof of Lemma 13.

Lemma 14. The random variables $o(|\sqrt{n}\, Y_n|)$ converge in probability to zero, as $n \to \infty$.

Proof: $E(P_n) = \pi_o$ and therefore by the weak law of large numbers, $P_n \to \pi_o$ in probability. That is, given $\varepsilon_1 > 0$ and $\delta_1 > 0$, there is an n_1 such that for $n > n_1$,

$$P(|P_n - \pi_o| < \delta_1) > 1 - \varepsilon_1 .$$

Similarly, $E(\sqrt{n}\; Y_{ni})^2 = 1 - \pi_{oi}$ so that

$$(\sqrt{n}\, Y_{ni})^2 \to 1 - \pi_{oi} \text{ in probability.}$$

(Recall that $Y_{ni} = \dfrac{P_{ni} - \pi_{oi}}{\sqrt{\pi_{oi}}}$ so that $E(\sqrt{n}\; Y_{ni})^2 = \dfrac{n}{\pi_{oi}}$ Var (P_{ni}).)

Since an elementary function of random variables, each of which converges (in probability) to a constant, converges (in probability) to the function of the constant, we have

$$|\sqrt{n}\ Y_n| = \sqrt{\sum (\sqrt{n}\ Y_{ni})^2} \rightarrow \sqrt{r-1} \quad \text{in probability.}$$

Thus for every $\varepsilon_2 > 0$ and $\delta_2 > 0$, there exists an n_2 such that

$$P\left(\Big|\,|\sqrt{n}\ Y_n| - \sqrt{r-1}\,\Big| < \delta_2\right) > 1 - \varepsilon_2, \quad \text{for} \quad n > n_2 .$$

Since $|\sqrt{n}\,Y_n| - \sqrt{r-1} \leq \big|\,|\sqrt{n}\ Y_n| - \sqrt{r-1}\,\big|$, we can write

$$P(|\sqrt{n}\ Y_n| < \delta_2 + \sqrt{r-1}\ ) > 1 - \varepsilon_2, \quad \text{for} \quad n > n_2 .$$

Now by definition of $o(|\sqrt{n}\ y|)$ as $p \rightarrow \pi_o$, $\lim\limits_{p \rightarrow \pi_o} \dfrac{o(|\sqrt{n}\ y|)}{|\sqrt{n}\ y|} = 0.$

Then for every $\varepsilon_3 > 0$, there is a $\delta_3 > 0$ such that whenever $|p - \pi_o| < \delta_3$, $o(|\sqrt{n}\ y|) < \varepsilon_3 |\sqrt{n}\ y|$.

We now complete the proof of the lemma by showing that for every $\varepsilon > 0$ and $\delta > 0$, there is an n_o such that $n > n_o$ implies that

$$P(o(|\sqrt{n}\ Y_n|) < \delta) > 1 - \varepsilon .$$

Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary, and choose

$$\varepsilon_3 = \frac{\delta}{1 + \sqrt{r-1}} .$$

Determine $\delta_3 > 0$ so that

$$o(|\sqrt{n}\ y|) < \varepsilon_3 |\sqrt{n}\ y| \quad \text{when} \quad |p - \pi_o| < \delta_3 .$$

Now take $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$, $\delta_1 = \delta_3$ and $\delta_2 = 1$. Let $n_o = \max\left\{n_1, n_2\right\}$ and consider $n > n_o$. Then

$$P(|P_n - \pi_o| < \delta_3) > 1 - \frac{\varepsilon}{2}$$

and

$$P(|\sqrt{n}\ Y_n| < 1 + \sqrt{r-1}) > 1 - \frac{\varepsilon}{2}.$$

Let A denote the event that $|P_n - \pi_o| < \delta_3$ and B denote the event that $|\sqrt{n}\ Y_n| < 1 + \sqrt{r-1}$. We have

$$\begin{aligned} P(A \cap B)' &= P(A' \cup B') \\ &\leq P(A') + P(B') \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon . \end{aligned}$$

Hence $P(A \cap B) > 1 - \varepsilon$.

Now let C be the event that $o(|\sqrt{n}\ Y_n| < \delta)$, and suppose that $A \cap B$ occurs. Then $|P_n - \pi_o| < \delta_3$ and $|\sqrt{n}\ Y_n| < 1 + \sqrt{r-1}$ so that

$$o(|\sqrt{n}\ Y_n|) < \frac{\delta}{1 + \sqrt{r-1}} |\sqrt{n}\ Y_n| < \delta .$$

Thus $(A \cap B) \subset C$ so that

$$P(C) \geq P(A \cap B) > 1 - \varepsilon .$$

Hence given $\varepsilon > 0$ and $\delta > 0$, there is an n_o so that

$$P(o(|\sqrt{n}\ Y_n|) < \delta) > 1 - \varepsilon \quad \text{for} \quad n > n_o.$$

This completes the proof of Lemma 14.

Corollary. The random variables $o(|\sqrt{n}\ Y_n|^2)$ converge in probability to zero, as $n \to \infty$.

Completion of the proof of the theorem:

From Lemma 4,

$$(5) \qquad \sqrt{n}\,(\hat{\theta}_n - \theta_o) = (A^T A)^{-1} A^T Z_n + o(|\sqrt{n}\ Y_n|)$$

(where $Z_n = \sqrt{n}\ Y_n$) and from Lemma 6,

$$(6) \qquad n \sum \frac{[P_{ni} - \pi_i(\hat{\theta}_n)]^2}{\pi_i(\hat{\theta}_n)} = Z_n^T Z_n - Z_n^T A(A^T A)^{-1} A^T Z_n + o(|\sqrt{n}\, Y_n|^2) \ .$$

Lemma 14 and its corollary give the limiting distribution of the $o(\ )$ terms and Lemma 13 gives the limiting distribution for the remainder of the right side of (5) and (6). The conclusion of the theorem then follows from Lemma 10.

This completes the proof of the theorem.

APPENDIX B

TABLES AND GRAPHS OF POWER FUNCTIONS

In this appendix some graphs and tables, of the power of the tests of the independence hypothesis for 2 x 2 contingency tables with no marginal totals fixed, are given. The tables are examples of the output from the Burroughs B5000 computer on which the calculations were made. The graphs were made using this output.

The tables are explained more fully in Chapter IV. P11 denotes p_{11}, P1. denotes $p_{1.}$ and P.1 denotes $p_{.1}$. IP11 is $p_{1.}p_{.1}$. UPX, UPL and UPE denote the power when $\alpha = 0.05$ using the χ^2 statistic, $-2 \log \lambda$ statistic and exact tests, respectively. Adding the suffix "U" denotes the corresponding power for $\alpha = 0.10$.

Two other sets of graphs and one other table are found in Chapter IV.

Table 2. Power for n = 20.

P1.	P.1	IP11	P11	UPX	UPL	UPE	UPXU	UPLU	UPEU
0.20	0.20	0.04	0.00	0.01962076	0.10972978	0.01227770	0.08712804	0.22692000	0.04134770
0.20	0.20	0.04	0.05	0.08032341	0.07686489	0.01388125	0.13218562	0.12516320	0.02434259
0.20	0.20	0.04	0.10	0.42877268	0.40162637	0.13534632	0.53422075	0.47571709	0.17196320
0.20	0.20	0.04	0.15	0.86900548	0.84260985	0.50901885	0.90211561	0.86938229	0.56552032
0.20	0.20	0.04	0.20	0.98847078	0.98847078	0.93082471	0.98847078	0.98847078	0.98847078
0.20	0.30	0.06	0.00	0.11696327	0.34461682	0.07665938	0.29896253	0.53274407	0.17724197
0.20	0.30	0.06	0.05	0.03163013	0.06993462	0.00975979	0.08495337	0.13533132	0.02665753
0.20	0.30	0.06	0.10	0.18868647	0.19350906	0.05785831	0.29022467	0.26653016	0.08545028
0.20	0.30	0.06	0.15	0.61073972	0.60601115	0.30197182	0.72314597	0.69322442	0.37667193
0.20	0.30	0.06	0.20	0.97329868	0.96306736	0.87151253	0.98377983	0.98389657	0.92585782
0.20	0.40	0.08	0.00	0.34707675	0.61728718	0.24313655	0.59162291	0.77839593	0.40531920
0.20	0.40	0.08	0.05	0.06240392	0.14264547	0.03355079	0.15392091	0.23242313	0.07457523
0.20	0.40	0.08	0.10	0.07555732	0.10535054	0.02722130	0.14997363	0.16834815	0.05015260
0.20	0.40	0.08	0.15	0.36160524	0.40823109	0.18930793	0.50465919	0.50760117	0.26983173
0.20	0.40	0.08	0.20	0.87486555	0.91511868	0.71112592	0.94544401	0.95478808	0.83615967
0.20	0.50	0.10	0.00	0.64637526	0.81440609	0.50079883	0.83116014	0.90199769	0.65152485
0.20	0.50	0.10	0.05	0.17251852	0.25846714	0.10206338	0.30708304	0.35709702	0.16766417
0.20	0.50	0.10	0.10	0.04266561	0.08225666	0.01720878	0.10750423	0.14328421	0.03852642
0.20	0.50	0.10	0.15	0.17251852	0.25846714	0.08673319	0.30708304	0.35709702	0.15916513
0.20	0.50	0.10	0.20	0.64637526	0.81440609	0.45642360	0.83116014	0.90199769	0.65152485
0.20	0.60	0.12	0.00	0.87486555	0.91511868	0.73518205	0.94544401	0.95478808	0.83615967
0.20	0.60	0.12	0.05	0.36160524	0.40823109	0.20639321	0.50465919	0.50760117	0.27965080
0.20	0.60	0.12	0.10	0.07555732	0.10535054	0.03037592	0.14997363	0.16834815	0.05265842
0.20	0.60	0.12	0.15	0.06240392	0.14264547	0.02827931	0.15392091	0.23242313	0.07147520
0.20	0.60	0.12	0.20	0.34707675	0.61728718	0.21605765	0.59162291	0.77839593	0.40531920
0.20	0.70	0.14	0.00	0.97329868	0.96306736	0.87326704	0.98377983	0.98389657	0.92585782
0.20	0.70	0.14	0.05	0.61073972	0.60601115	0.30817276	0.72314597	0.69322442	0.38043307
0.20	0.70	0.14	0.10	0.18868647	0.19350906	0.06030222	0.29022467	0.26653016	0.08762244
0.20	0.70	0.14	0.15	0.03163013	0.06993462	0.00928520	0.08495337	0.13533132	0.02631734
0.20	0.70	0.14	0.20	0.11696327	0.34461682	0.07042898	0.29896253	0.53274407	0.17724197
0.20	0.80	0.16	0.00	0.98847078	0.98847078	0.93082471	0.98847078	0.98847078	0.98847078
0.20	0.80	0.16	0.05	0.86900548	0.84260985	0.50939046	0.90211561	0.86938229	0.56566812
0.20	0.80	0.16	0.10	0.42877268	0.40162637	0.13601091	0.53422075	0.47571709	0.17254428
0.20	0.80	0.16	0.15	0.08032341	0.07686489	0.01395800	0.13218562	0.12516320	0.02441082
0.20	0.80	0.16	0.20	0.01962076	0.10972978	0.01168454	0.08712804	0.22692000	0.04134770
0.30	0.20	0.06	0.00	0.11696327	0.34461682	0.07665938	0.29896253	0.53274407	0.17724197
0.30	0.20	0.06	0.05	0.03163013	0.06993462	0.00975979	0.08495337	0.13533132	0.02665753
0.30	0.20	0.06	0.10	0.18868647	0.19350906	0.05785831	0.29022467	0.26653016	0.08545028
0.30	0.20	0.06	0.15	0.61073972	0.60601115	0.30197182	0.72314597	0.69322442	0.37667193
0.30	0.20	0.06	0.20	0.97329868	0.96306736	0.87151253	0.98377983	0.98389657	0.92585782
0.30	0.30	0.09	0.00	0.39888338	0.70816630	0.31164130	0.65986726	0.84720508	0.50843603
0.30	0.30	0.09	0.05	0.08237378	0.18644650	0.05291625	0.19704803	0.28199923	0.11658124
0.30	0.30	0.09	0.10	0.05645838	0.07846976	0.01705950	0.12340013	0.13314505	0.03389707
0.30	0.30	0.09	0.15	0.25732029	0.26293921	0.08899627	0.38630324	0.35199735	0.12862180
0.30	0.30	0.09	0.20	0.65978770	0.65930315	0.32107059	0.78011159	0.74246463	0.40987295
0.30	0.30	0.09	0.25	0.95652335	0.95282136	0.70757661	0.97812274	0.96810831	0.78372107
0.30	0.30	0.09	0.30	0.99920208	0.99920208	0.99236274	0.99920208	0.99920208	0.99920208
0.30	0.40	0.12	0.00	0.74433611	0.91839120	0.64610333	0.90344577	0.96797925	0.80528256
0.30	0.40	0.12	0.05	0.25467739	0.36608255	0.18196287	0.42246499	0.47556384	0.29949245
0.30	0.40	0.12	0.10	0.05841790	0.09994934	0.03214469	0.13565426	0.16218438	0.06991025
0.30	0.40	0.12	0.15	0.09422410	0.11196402	0.03608714	0.17928516	0.17914175	0.06484081
0.30	0.40	0.12	0.20	0.35667079	0.37495012	0.17535179	0.50297442	0.48586533	0.25432891
0.30	0.40	0.12	0.25	0.77017476	0.78230130	0.52153700	0.86704081	0.85491785	0.62813338
0.30	0.40	0.12	0.30	0.99561679	0.99535150	0.97389683	0.99826895	0.99834596	0.98929199
0.30	0.50	0.15	0.00	0.94636630	0.98254532	0.89202755	0.98564354	0.99292908	0.94950053
0.30	0.50	0.15	0.05	0.50934676	0.56885829	0.38074049	0.66673739	0.67580947	0.50283015
0.30	0.50	0.15	0.10	0.15618790	0.19498952	0.09565319	0.27223852	0.28348608	0.16127471
0.30	0.50	0.15	0.15	0.04958207	0.07379083	0.02279779	0.11543007	0.12793025	0.05026645
0.30	0.50	0.15	0.20	0.15618790	0.19498952	0.08033100	0.27223852	0.28348608	0.14319281
0.30	0.50	0.15	0.25	0.50934675	0.56885829	0.34377550	0.66673739	0.67580947	0.47487946
0.30	0.50	0.15	0.30	0.94636630	0.98254532	0.87146927	0.98564354	0.99292908	0.94950053
0.30	0.60	0.18	0.00	0.99561679	0.99535150	0.97582679	0.99826895	0.99834596	0.98929199
0.30	0.60	0.18	0.05	0.77017476	0.78230130	0.54682454	0.86704082	0.85491785	0.64258162
0.30	0.60	0.18	0.10	0.35667079	0.37495012	0.19696540	0.50297442	0.48586533	0.27456992
0.30	0.60	0.18	0.15	0.09422410	0.11196402	0.04201164	0.17928516	0.17914175	0.07291359
0.30	0.60	0.18	0.20	0.05841790	0.09994934	0.02843334	0.13565426	0.16218438	0.06470183
0.30	0.60	0.18	0.25	0.25467739	0.36608255	0.16272909	0.42246499	0.47556384	0.28210572
0.30	0.60	0.18	0.30	0.74433611	0.91839120	0.61780456	0.90344577	0.96797925	0.80528256
0.30	0.70	0.21	0.00	0.99920208	0.99920208	0.99236274	0.99920208	0.99920208	0.99928208
0.30	0.70	0.21	0.05	0.95652335	0.95282136	0.71061568	0.97812274	0.96810831	0.78450903
0.30	0.70	0.21	0.10	0.65978770	0.65930315	0.33273226	0.78011159	0.74246463	0.41708326
0.30	0.70	0.21	0.15	0.25732029	0.26293921	0.09602779	0.38630324	0.35199735	0.13638430
0.30	0.70	0.21	0.20	0.05645838	0.07846976	0.01787576	0.12340013	0.13314505	0.03512139
0.30	0.70	0.21	0.25	0.08237378	0.18644650	0.04861384	0.19704803	0.28199923	0.11210664
0.30	0.70	0.21	0.30	0.39888338	0.70816630	0.29643914	0.65986726	0.84720508	0.50843603
0.30	0.80	0.24	0.10	0.97329868	0.96306736	0.87326704	0.98377983	0.98389657	0.92585782
0.30	0.80	0.24	0.15	0.61073972	0.60601115	0.30817276	0.72314597	0.69322442	0.38043307
0.30	0.80	0.24	0.20	0.18868647	0.19350906	0.06030222	0.29022467	0.26653016	0.08762244
0.30	0.80	0.24	0.25	0.03163013	0.06993462	0.00928520	0.08495337	0.13533132	0.02631734
0.30	0.80	0.24	0.30	0.11696327	0.34461682	0.07042898	0.29896253	0.53274407	0.17724197
0.40	0.20	0.08	0.00	0.34707675	0.61728718	0.24313655	0.59162291	0.77839593	0.40531920
0.40	0.20	0.08	0.05	0.06240392	0.14264547	0.03355079	0.15392091	0.23242313	0.07457523
0.40	0.20	0.08	0.10	0.07555732	0.10535054	0.02722130	0.14997363	0.16834815	0.05015260
0.40	0.20	0.08	0.15	0.36160524	0.40823109	0.18930793	0.50465919	0.50760117	0.26983173
0.40	0.20	0.08	0.20	0.87486555	0.91511868	0.71112592	0.94544401	0.95478808	0.83615967
0.40	0.30	0.12	0.00	0.74433611	0.91839120	0.64610333	0.90344577	0.96797925	0.80528256

Table 2. (Continued)

0.40	0.30	0.12	0.05	0.25467739	0.36608255	0.18196287	0.42246499	0.47556384	0.29949245
0.40	0.30	0.12	0.10	0.05841790	0.09994934	0.03214469	0.13565426	0.16218438	0.06991025
0.40	0.30	0.12	0.15	0.09422410	0.11196402	0.03608714	0.17928516	0.17914175	0.06484081
0.40	0.30	0.12	0.20	0.35667079	0.37495012	0.17535179	0.50297442	0.48586533	0.25432891
0.40	0.30	0.12	0.25	0.77017476	0.78230130	0.52153700	0.86704081	0.85491785	0.62813338
0.40	0.30	0.12	0.30	0.99561679	0.99535150	0.97389683	0.99826895	0.99834596	0.98929199
0.40	0.40	0.16	0.00	0.96112342	0.99311036	0.92719279	0.99192840	0.99802264	0.97349767
0.40	0.40	0.16	0.05	0.55875929	0.61691195	0.44974479	0.71764917	0.72807714	0.59552783
0.40	0.40	0.16	0.10	0.19022472	0.23001655	0.13061444	0.32303733	0.33463302	0.22498689
0.40	0.40	0.16	0.15	0.05341021	0.07112985	0.02764748	0.12333934	0.12991794	0.06267942
0.40	0.40	0.16	0.20	0.11527863	0.12597456	0.04754007	0.21389446	0.21060600	0.09383075
0.40	0.40	0.16	0.25	0.38522716	0.39564449	0.19795097	0.54181245	0.52933262	0.31092954
0.40	0.40	0.16	0.30	0.77164028	0.77675299	0.51642898	0.87492332	0.86350515	0.65634225
0.40	0.40	0.16	0.35	0.98135521	0.98143188	0.85510516	0.99348258	0.99115702	0.91924133
0.40	0.40	0.16	0.40	0.99996343	0.99996343	0.99947561	0.99996343	0.99996343	0.99996343
0.40	0.50	0.20	0.00	0.99897582	0.99954969	0.99543433	0.99980775	0.99985447	0.99854690
0.40	0.50	0.20	0.05	0.83468846	0.84782931	0.70666378	0.91742526	0.91493806	0.81006528
0.40	0.50	0.20	0.10	0.44805309	0.46811108	0.32439183	0.60905678	0.60623679	0.45606260
0.40	0.50	0.20	0.15	0.14519970	0.16025014	0.09133386	0.25899217	0.25968963	0.16361261
0.40	0.50	0.20	0.20	0.05118539	0.06197141	0.02444557	0.11928057	0.12186114	0.05857943
0.40	0.50	0.20	0.25	0.14519970	0.16025014	0.06997999	0.25899217	0.25968963	0.13964878
0.40	0.50	0.20	0.30	0.44805309	0.46811108	0.27287127	0.60905678	0.60623679	0.41675178
0.40	0.50	0.20	0.35	0.83468846	0.84782931	0.66148237	0.91742526	0.91493806	0.78817007
0.40	0.50	0.20	0.40	0.99897582	0.99954969	0.99421009	0.99980775	0.99985447	0.99854690
0.40	0.60	0.24	0.00	0.99996343	0.99996343	0.99947561	0.99996343	0.99996343	0.99996343
0.40	0.60	0.24	0.05	0.98135521	0.98143188	0.86323331	0.99348258	0.99115702	0.92083090
0.40	0.60	0.24	0.10	0.77164028	0.77675299	0.55778054	0.87492332	0.86350515	0.67485749
0.40	0.60	0.24	0.15	0.38522716	0.39564449	0.23609016	0.54181245	0.52933262	0.34083419
0.40	0.60	0.24	0.20	0.11527863	0.12597456	0.06065699	0.21389446	0.21060600	0.10979494
0.40	0.60	0.24	0.25	0.05341021	0.07112985	0.02486813	0.12333934	0.12991794	0.05865038
0.40	0.60	0.24	0.30	0.19022472	0.23001655	0.10835927	0.32303733	0.33463302	0.19932004
0.40	0.60	0.24	0.35	0.55875929	0.61691195	0.40621792	0.71764917	0.72807714	0.56199927
0.40	0.60	0.24	0.40	0.96112342	0.99311036	0.91386164	0.99192840	0.99802264	0.97349767
0.40	0.70	0.28	0.10	0.99561679	0.99535150	0.97582679	0.99826895	0.99834596	0.98929199
0.40	0.70	0.28	0.15	0.77017476	0.78230130	0.54682454	0.86704082	0.85491785	0.64258162
0.40	0.70	0.28	0.20	0.35667079	0.37495012	0.19696540	0.50297442	0.48586533	0.27456992
0.40	0.70	0.28	0.25	0.09422410	0.11196402	0.04201164	0.17928516	0.17914175	0.07291359
0.40	0.70	0.28	0.30	0.05841790	0.09994934	0.02843334	0.13565426	0.16218438	0.06470183
0.40	0.70	0.28	0.35	0.25467739	0.36608255	0.16272909	0.42246499	0.47556384	0.28210572
0.40	0.70	0.28	0.40	0.74433611	0.91839120	0.61780456	0.90344577	0.96797925	0.80528256
0.40	0.80	0.32	0.20	0.87486555	0.91511868	0.73518205	0.94544401	0.95478808	0.83615967
0.40	0.80	0.32	0.25	0.36160524	0.40823109	0.20639321	0.50465919	0.50760117	0.27965080
0.40	0.80	0.32	0.30	0.07555732	0.10535054	0.03037592	0.14997363	0.16834815	0.05265842
0.40	0.80	0.32	0.35	0.06240392	0.14264547	0.02827931	0.15392091	0.23242313	0.07147520
0.40	0.80	0.32	0.40	0.34707675	0.61728718	0.21605765	0.59162291	0.77839593	0.40531920
0.50	0.20	0.10	0.00	0.64637526	0.81440609	0.50079883	0.83116014	0.90199769	0.65152485
0.50	0.20	0.10	0.05	0.17251852	0.25846714	0.10206338	0.30708304	0.35709702	0.16766417
0.50	0.20	0.10	0.10	0.04266561	0.08225666	0.01720878	0.10750423	0.14328421	0.03852642
0.50	0.20	0.10	0.15	0.17251852	0.25846714	0.08673319	0.30708304	0.35709702	0.15916513
0.50	0.20	0.10	0.20	0.64637526	0.81440609	0.45642360	0.83116014	0.90199769	0.65152485
0.50	0.30	0.15	0.00	0.94636630	0.98254532	0.89202755	0.98564354	0.99292908	0.94950053
0.50	0.30	0.15	0.05	0.50934676	0.56885829	0.38074049	0.66673739	0.67580947	0.50283015
0.50	0.30	0.15	0.10	0.15618790	0.19498952	0.09565319	0.27223852	0.28348608	0.16127471
0.50	0.30	0.15	0.15	0.04958207	0.07379083	0.02279779	0.11543007	0.12793025	0.05026645
0.50	0.30	0.15	0.20	0.15618790	0.19498952	0.08033100	0.27223852	0.28348608	0.14319281
0.50	0.30	0.15	0.25	0.50934675	0.56885829	0.34377550	0.66673739	0.67580947	0.47487946
0.50	0.30	0.15	0.30	0.94636630	0.98254532	0.87146927	0.98564354	0.99292908	0.94950053
0.50	0.40	0.20	0.00	0.99897582	0.99954969	0.99543433	0.99980775	0.99985447	0.99854690
0.50	0.40	0.20	0.05	0.83468846	0.84782931	0.70666378	0.91742526	0.91493806	0.81006528
0.50	0.40	0.20	0.10	0.44805309	0.46811108	0.32439183	0.60905678	0.60623679	0.45606260
0.50	0.40	0.20	0.15	0.14519970	0.16025014	0.09133386	0.25899217	0.25968963	0.16361261
0.50	0.40	0.20	0.20	0.05118539	0.06197141	0.02444557	0.11928057	0.12186114	0.05857943
0.50	0.40	0.20	0.25	0.14519970	0.16025014	0.06997999	0.25899217	0.25968963	0.13964878
0.50	0.40	0.20	0.30	0.44805309	0.46811108	0.27287127	0.60905678	0.60623679	0.41675178
0.50	0.40	0.20	0.35	0.83468846	0.84782931	0.66148237	0.91742526	0.91493806	0.78817007
0.50	0.40	0.20	0.40	0.99897582	0.99954969	0.99421009	0.99980775	0.99985447	0.99854690
0.50	0.50	0.25	0.00	0.99999809	0.99999809	0.99995995	0.99999809	0.99999809	0.99999809
0.50	0.50	0.25	0.05	0.98691225	0.98729471	0.92257404	0.99615817	0.99546145	0.96361258
0.50	0.50	0.25	0.10	0.80442095	0.80968571	0.66048169	0.90086652	0.89755514	0.77994853
0.50	0.50	0.25	0.15	0.42967464	0.43982832	0.31213493	0.59480854	0.59180546	0.44770214
0.50	0.50	0.25	0.20	0.14112318	0.14960067	0.09119940	0.25569780	0.25522846	0.16671154
0.50	0.50	0.25	0.25	0.05120080	0.05754776	0.02483386	0.12046459	0.12121701	0.06235089
0.50	0.50	0.25	0.30	0.14112318	0.14960067	0.06573417	0.25569780	0.25522846	0.14068927
0.50	0.50	0.25	0.35	0.42967464	0.43982832	0.25043542	0.59480854	0.59180546	0.40603839
0.50	0.50	0.25	0.40	0.80442095	0.80968571	0.60055108	0.90086652	0.89755514	0.75620277
0.50	0.50	0.25	0.45	0.98691225	0.98729471	0.91165695	0.99615817	0.99546145	0.96168262
0.50	0.50	0.25	0.50	0.99999809	0.99999809	0.99995995	0.99999809	0.99999809	0.99999809
0.50	0.60	0.30	0.10	0.99897582	0.99954969	0.99543433	0.99980775	0.99985447	0.99854690
0.50	0.60	0.30	0.15	0.83468846	0.84782931	0.70666378	0.91742526	0.91493806	0.81006528
0.50	0.60	0.30	0.20	0.44805309	0.46811108	0.32439183	0.60905678	0.60623679	0.45606260
0.50	0.60	0.30	0.25	0.14519970	0.16025014	0.09133386	0.25899217	0.25968963	0.16361261
0.50	0.60	0.30	0.30	0.05118539	0.06197141	0.02444557	0.11928057	0.12186114	0.05857943
0.50	0.60	0.30	0.35	0.14519970	0.16025014	0.06997999	0.25899217	0.25968963	0.13964878
0.50	0.60	0.30	0.40	0.44805309	0.46811108	0.27287127	0.60905678	0.60623679	0.41675178
0.50	0.60	0.30	0.45	0.83468846	0.84782931	0.66148237	0.91742526	0.91493806	0.78817007
0.50	0.60	0.30	0.50	0.99897582	0.99954969	0.99421009	0.99980775	0.99985447	0.99854690

Table 2. (Continued)

0.50	0.70	0.35	0.20	0.94636630	0.98254532	0.89202755	0.98564354	0.99292908	0.94950053
0.50	0.70	0.35	0.25	0.50934676	0.56885829	0.38074049	0.66673739	0.67580947	0.50283015
0.50	0.70	0.35	0.30	0.15618790	0.19498952	0.09565319	0.27223852	0.28348608	0.16127471
0.50	0.70	0.35	0.35	0.04958207	0.07379083	0.02279779	0.11543007	0.12793025	0.05026645
0.50	0.70	0.35	0.40	0.15618790	0.19498952	0.08033100	0.27223852	0.28348608	0.14319281
0.50	0.70	0.35	0.45	0.50934676	0.56885829	0.34377550	0.66673739	0.67580947	0.47487946
0.50	0.70	0.35	0.50	0.94636630	0.98254532	0.87146927	0.98564354	0.99292908	0.94950053
0.50	0.80	0.40	0.30	0.64637526	0.81440609	0.50079883	0.83116014	0.90199769	0.65152485
0.50	0.80	0.40	0.35	0.17251852	0.25846714	0.10206338	0.30708304	0.35709702	0.16766417
0.50	0.80	0.40	0.40	0.04266561	0.08225666	0.01720878	0.10750423	0.14328421	0.03852642
0.50	0.80	0.40	0.45	0.17251852	0.25846714	0.08673319	0.30708304	0.35709702	0.15916513
0.50	0.80	0.40	0.50	0.64637526	0.81440609	0.45642360	0.83116014	0.90199769	0.65152485
0.60	0.20	0.12	0.00	0.87486555	0.91511868	0.73518205	0.94544401	0.95478808	0.83615967
0.60	0.20	0.12	0.05	0.36160524	0.40823109	0.20639321	0.50465919	0.50760117	0.27965080
0.60	0.20	0.12	0.10	0.07555732	0.10535054	0.03037592	0.14997363	0.16834815	0.05268842
0.60	0.20	0.12	0.15	0.06240392	0.14264547	0.02827931	0.15392091	0.23242313	0.07147520
0.60	0.20	0.12	0.20	0.34707675	0.61728718	0.21605765	0.59162291	0.77839593	0.40531920
0.60	0.30	0.18	0.00	0.99561679	0.99535150	0.97582679	0.99826895	0.99834596	0.98929199
0.60	0.30	0.18	0.05	0.77017476	0.78230130	0.54682454	0.86704082	0.85491785	0.64258162
0.60	0.30	0.18	0.10	0.35667079	0.37495012	0.19696540	0.50297442	0.48586533	0.27456992
0.60	0.30	0.18	0.15	0.09422410	0.11196402	0.04201164	0.17928516	0.17914175	0.07291359
0.60	0.30	0.18	0.20	0.05841790	0.09994934	0.02843334	0.13565426	0.16218438	0.06470183
0.60	0.30	0.18	0.25	0.25467739	0.36608255	0.16272909	0.42246499	0.47556384	0.28210572
0.60	0.30	0.18	0.30	0.74433611	0.91839120	0.61780456	0.90344577	0.96797925	0.80528256
0.60	0.40	0.24	0.00	0.99996343	0.99996343	0.99947561	0.99996343	0.99996343	0.99996343
0.60	0.40	0.24	0.05	0.98135521	0.98143188	0.86323331	0.99348258	0.99115702	0.92083090
0.60	0.40	0.24	0.10	0.77164028	0.77675299	0.55778054	0.87492332	0.86350515	0.67485749
0.60	0.40	0.24	0.15	0.38522716	0.39564449	0.23609016	0.54181245	0.52933262	0.34083419
0.60	0.40	0.24	0.20	0.11527863	0.12597456	0.06065699	0.21389446	0.21060600	0.10979494
0.60	0.40	0.24	0.25	0.05341021	0.07112985	0.02486813	0.12333934	0.12991794	0.05865038
0.60	0.40	0.24	0.30	0.19022472	0.23001655	0.10835927	0.32303733	0.33463302	0.19932004
0.60	0.40	0.24	0.35	0.55875929	0.61691195	0.40621792	0.71764917	0.72807714	0.56199927
0.60	0.40	0.24	0.40	0.96112342	0.99311036	0.91386164	0.99192840	0.99802264	0.97349767
0.60	0.50	0.30	0.10	0.99897582	0.99954969	0.99543433	0.99980775	0.99985447	0.99854690
0.60	0.50	0.30	0.15	0.83468846	0.84782931	0.70666378	0.91742526	0.91493806	0.81006528
0.60	0.50	0.30	0.20	0.44805309	0.46811108	0.32439183	0.60905678	0.60623679	0.45606260
0.60	0.50	0.30	0.25	0.14519970	0.16025014	0.09133386	0.25899217	0.25968963	0.16361261
0.60	0.50	0.30	0.30	0.05118539	0.06197141	0.02444557	0.11928057	0.12186114	0.05857943
0.60	0.50	0.30	0.35	0.14519970	0.16025014	0.06997999	0.25899217	0.25968963	0.13964878
0.60	0.50	0.30	0.40	0.44805309	0.46811108	0.27287127	0.60905678	0.60623679	0.41675178
0.60	0.50	0.30	0.45	0.83468846	0.84782931	0.66148237	0.91742526	0.91493806	0.78817007
0.60	0.50	0.30	0.50	0.99897582	0.99954969	0.99421009	0.99980775	0.99985447	0.99854690
0.60	0.60	0.36	0.20	0.96112342	0.99311036	0.92719279	0.99192840	0.99802264	0.97349767
0.60	0.60	0.36	0.25	0.55875929	0.61691195	0.44974479	0.71764917	0.72807714	0.59552783
0.60	0.60	0.36	0.30	0.19022472	0.23001655	0.13061444	0.32303733	0.33463302	0.22498689
0.60	0.60	0.36	0.35	0.05341021	0.07112985	0.02764748	0.12333934	0.12991794	0.06267942
0.60	0.60	0.36	0.40	0.11527863	0.12597456	0.04754007	0.21389446	0.21060600	0.09383075
0.60	0.60	0.36	0.45	0.38522716	0.39564449	0.19795097	0.54181245	0.52933262	0.31092954
0.60	0.60	0.36	0.50	0.77164028	0.77675299	0.51642898	0.87492332	0.86350515	0.65634225
0.60	0.60	0.36	0.55	0.98135521	0.98143188	0.85510516	0.99348258	0.99115702	0.91924133
0.60	0.60	0.36	0.60	0.99996343	0.99996343	0.99947561	0.99996343	0.99996343	0.99996343
0.60	0.70	0.42	0.30	0.74433611	0.91839120	0.64610333	0.90344577	0.96797925	0.80528256
0.60	0.70	0.42	0.35	0.25467739	0.36608255	0.18196287	0.42246499	0.47556384	0.29949245
0.60	0.70	0.42	0.40	0.05841790	0.09994934	0.03214469	0.13565426	0.16218438	0.06991025
0.60	0.70	0.42	0.45	0.09422410	0.11196402	0.03608714	0.17928516	0.17914175	0.06484081
0.60	0.70	0.42	0.50	0.35667079	0.37495012	0.17535179	0.50297442	0.48586533	0.25432891
0.60	0.70	0.42	0.55	0.77017476	0.78230130	0.52153700	0.86704081	0.85491785	0.62813338
0.60	0.70	0.42	0.60	0.99561679	0.99535150	0.97389683	0.99826895	0.99834596	0.98929199
0.60	0.80	0.48	0.40	0.34707675	0.61728718	0.24313655	0.59162291	0.77839593	0.40531920
0.60	0.80	0.48	0.45	0.06240392	0.14264547	0.03355079	0.15392091	0.23242313	0.07457523
0.60	0.80	0.48	0.50	0.07555732	0.10535054	0.02722130	0.14997363	0.16834815	0.05015260
0.60	0.80	0.48	0.55	0.36160524	0.40823109	0.18930793	0.50465919	0.50760117	0.26983173
0.60	0.80	0.48	0.60	0.87486555	0.91511868	0.71112592	0.94544401	0.95478808	0.83615967
0.70	0.20	0.14	0.00	0.97329868	0.96306736	0.87326704	0.98377983	0.98389657	0.92585782
0.70	0.20	0.14	0.05	0.61073972	0.60601115	0.30817276	0.72314597	0.69322442	0.38043307
0.70	0.20	0.14	0.10	0.18868647	0.19350906	0.06030222	0.29022467	0.26653016	0.08762244
0.70	0.20	0.14	0.15	0.03163013	0.06993462	0.00928520	0.08495337	0.13533132	0.02631734
0.70	0.20	0.14	0.20	0.11696327	0.34461682	0.07042898	0.29896253	0.53274407	0.17724197
0.70	0.30	0.21	0.00	0.99920208	0.99920208	0.99236274	0.99920208	0.99920208	0.99920208
0.70	0.30	0.21	0.05	0.95652335	0.95282136	0.71061568	0.97812274	0.96810831	0.78458903
0.70	0.30	0.21	0.10	0.65978770	0.65930315	0.33273226	0.78011159	0.74246463	0.41708326
0.70	0.30	0.21	0.15	0.25732029	0.26293921	0.09602779	0.38630324	0.35199735	0.13638430
0.70	0.30	0.21	0.20	0.05645838	0.07846976	0.01787576	0.12340013	0.13314505	0.03512139
0.70	0.30	0.21	0.25	0.08237378	0.18644650	0.04861384	0.19704803	0.28199923	0.11210664
0.70	0.30	0.21	0.30	0.39888338	0.70816630	0.29643914	0.65986726	0.84720508	0.50843603
0.70	0.40	0.28	0.10	0.99561679	0.99535150	0.97582679	0.99826895	0.99834596	0.98929199
0.70	0.40	0.28	0.15	0.77017476	0.78230130	0.54682454	0.86704081	0.85491785	0.64258162
0.70	0.40	0.28	0.20	0.35667079	0.37495012	0.19696540	0.50297442	0.48586533	0.27456992
0.70	0.40	0.28	0.25	0.09422410	0.11196402	0.04201164	0.17928516	0.17914175	0.07291359
0.70	0.40	0.28	0.30	0.05841790	0.09994934	0.02843334	0.13565426	0.16218438	0.06470183
0.70	0.40	0.28	0.35	0.25467739	0.36608255	0.16272909	0.42246499	0.47556384	0.28210572
0.70	0.40	0.28	0.40	0.74433611	0.91839120	0.61780456	0.90344577	0.96797925	0.80528256

Table 2. (Concluded)

0.70	0.50	0.35	0.20	0.94636630	0.98254532	0.89202755	0.98564354	0.99292908	0.94950053
0.70	0.50	0.35	0.25	0.50934676	0.56885829	0.38074049	0.66673739	0.67580947	0.50283015
0.70	0.50	0.35	0.30	0.15618790	0.19498952	0.09565319	0.27223852	0.28348608	0.16127471
0.70	0.50	0.35	0.35	0.04958207	0.07379083	0.02279779	0.11543007	0.12793025	0.05026645
0.70	0.50	0.35	0.40	0.15618790	0.19498952	0.08033100	0.27223852	0.28348608	0.14319281
0.70	0.50	0.35	0.45	0.50934676	0.56885829	0.34377550	0.66673739	0.67580947	0.47487946
0.70	0.50	0.35	0.50	0.94636630	0.98254532	0.87146927	0.98564354	0.99292908	0.94950053
0.70	0.60	0.42	0.30	0.74433611	0.91839120	0.64610333	0.90344577	0.96797925	0.80528256
0.70	0.60	0.42	0.35	0.25467739	0.36608255	0.18196287	0.42246499	0.47556384	0.29949245
0.70	0.60	0.42	0.40	0.05841790	0.09994934	0.03214469	0.13565426	0.16218438	0.06991025
0.70	0.60	0.42	0.45	0.09422410	0.11196402	0.03608714	0.17928516	0.17914175	0.06484081
0.70	0.60	0.42	0.50	0.35667079	0.37495012	0.17535179	0.50297442	0.48586533	0.25432891
0.70	0.60	0.42	0.55	0.77017476	0.78230130	0.52153700	0.86704081	0.85491785	0.62813338
0.70	0.60	0.42	0.60	0.99561679	0.99535150	0.97389683	0.99826895	0.99834596	0.98929199
0.70	0.70	0.49	0.40	0.39888338	0.70816630	0.31164130	0.65986726	0.84720508	0.50843603
0.70	0.70	0.49	0.45	0.08237378	0.18644650	0.05291625	0.19704803	0.28199923	0.11658124
0.70	0.70	0.49	0.50	0.05645838	0.07846976	0.01705950	0.12340013	0.13314505	0.03389707
0.70	0.70	0.49	0.55	0.25732029	0.26293921	0.08899627	0.38630324	0.35199735	0.12862180
0.70	0.70	0.49	0.60	0.65978770	0.65930315	0.32107059	0.78011159	0.74246463	0.40987295
0.70	0.70	0.49	0.65	0.95652335	0.95282136	0.70757661	0.97812274	0.96810831	0.78372107
0.70	0.70	0.49	0.70	0.99920208	0.99920208	0.99236274	0.99920208	0.99920208	0.99920208
0.70	0.80	0.56	0.50	0.11696327	0.34461682	0.07665938	0.29896253	0.53274407	0.17724197
0.70	0.80	0.56	0.55	0.03163013	0.06993462	0.00975979	0.08495337	0.13533132	0.02665753
0.70	0.80	0.56	0.60	0.18868647	0.19350906	0.05785831	0.29022467	0.26653016	0.08545028
0.70	0.80	0.56	0.65	0.61073972	0.60601115	0.30197182	0.72314597	0.69322442	0.37667193
0.70	0.80	0.56	0.70	0.97329868	0.96306736	0.87151253	0.98377983	0.98389657	0.92585782
0.80	0.20	0.16	0.00	0.98847078	0.98847078	0.93082471	0.98847078	0.98847078	0.98847078
0.80	0.20	0.16	0.05	0.86900548	0.84260985	0.50939046	0.90211561	0.86938229	0.56566812
0.80	0.20	0.16	0.10	0.42877268	0.40162637	0.13601091	0.53422075	0.47571709	0.17254428
0.80	0.20	0.16	0.15	0.08032341	0.07686489	0.01395800	0.13218562	0.12516320	0.02441082
0.80	0.20	0.16	0.20	0.00000001	0.00000001	0.00000001	0.00000001	0.00000001	0.00000001
0.80	0.30	0.24	0.10	0.97329868	0.96306736	0.87326704	0.98377983	0.98389657	0.92585782
0.80	0.30	0.24	0.15	0.61073972	0.60601115	0.30817276	0.72314597	0.69322442	0.38043307
0.80	0.30	0.24	0.20	0.18868647	0.19350906	0.06030222	0.29022467	0.26653016	0.08762244
0.80	0.30	0.24	0.25	0.03163013	0.06993462	0.00928520	0.08495337	0.13533132	0.02631734
0.80	0.30	0.24	0.30	0.11696327	0.34461682	0.07042898	0.29896253	0.53274407	0.17724197
0.80	0.40	0.32	0.20	0.87486555	0.91511868	0.73518205	0.94544401	0.95478808	0.83615967
0.80	0.40	0.32	0.25	0.36160524	0.40823109	0.20639321	0.50465919	0.50760117	0.27965080
0.80	0.40	0.32	0.30	0.07555732	0.10535054	0.03037592	0.14997363	0.16834815	0.05265842
0.80	0.40	0.32	0.35	0.06240392	0.14264547	0.02827931	0.15392091	0.23242313	0.07147520
0.80	0.40	0.32	0.40	0.34707675	0.61728718	0.21605765	0.59162291	0.77839593	0.40531920
0.80	0.50	0.40	0.30	0.64637526	0.81440609	0.50079883	0.83116014	0.90199769	0.65152485
0.80	0.50	0.40	0.35	0.17251852	0.25846714	0.10206338	0.30708304	0.35709702	0.16766417
0.80	0.50	0.40	0.40	0.04266561	0.08225666	0.01720878	0.10750423	0.14328421	0.03852642
0.80	0.50	0.40	0.45	0.17251852	0.25846714	0.08673319	0.30708304	0.35709702	0.15916513
0.80	0.50	0.40	0.50	0.64637526	0.81440609	0.45642360	0.83116014	0.90199769	0.65152485
0.80	0.60	0.48	0.40	0.34707675	0.61728718	0.24313655	0.59162291	0.77839593	0.40531920
0.80	0.60	0.48	0.45	0.06240392	0.14264547	0.03355079	0.15392091	0.23242313	0.07457523
0.80	0.60	0.48	0.50	0.07555732	0.10535054	0.02722130	0.14997363	0.16834815	0.05015260
0.80	0.60	0.48	0.55	0.36160524	0.40823109	0.18930793	0.50465919	0.50760117	0.26983173
0.80	0.60	0.48	0.60	0.87486555	0.91511868	0.71112592	0.94544401	0.95478808	0.83615967
0.80	0.70	0.56	0.50	0.11696327	0.34461682	0.07665938	0.29896253	0.53274407	0.17724197
0.80	0.70	0.56	0.55	0.03163013	0.06993462	0.00975979	0.08495337	0.13533132	0.02665753
0.80	0.70	0.56	0.60	0.18868647	0.19350906	0.05785831	0.29022467	0.26653016	0.08545028
0.80	0.70	0.56	0.65	0.61073972	0.60601115	0.30197182	0.72314597	0.69322442	0.37667193
0.80	0.70	0.56	0.70	0.97329868	0.96306736	0.87151253	0.98377983	0.98389657	0.92585782
0.80	0.80	0.64	0.60	0.01962076	0.10972978	0.01227770	0.08712804	0.22692000	0.04134770
0.80	0.80	0.64	0.65	0.08032341	0.07686489	0.01388125	0.13218562	0.12516320	0.02434259
0.80	0.80	0.64	0.70	0.42877268	0.40162637	0.13534632	0.53422075	0.47571709	0.17196320
0.80	0.80	0.64	0.75	0.86900548	0.84260985	0.50901885	0.90211561	0.86938229	0.56552032
0.80	0.80	0.64	0.80	0.98847078	0.98847078	0.93082471	0.98847078	0.98847078	0.98847078

Table 3. Power for n = 50.

P1.	P.1	IP11	P11	UPX	UPL	UPE	UPXU	UPLU	UPEU
0.20	0.20	0.04	0.00	0.29365989	0.70261136	0.29358084	0.57583625	0.88623181	0.51285607
0.20	0.20	0.04	0.05	0.08618949	0.08439153	0.00368147	0.14688081	0.16224171	0.01142575
0.20	0.20	0.04	0.10	0.71938752	0.67156313	0.02388131	0.79901417	0.78154860	0.03803735
0.20	0.20	0.04	0.15	0.99399003	0.98952707	0.28697455	0.99675405	0.99498856	0.36293700
0.20	0.20	0.04	0.20	0.99998573	0.99998573	0.99998573	0.99998573	0.99998573	0.99998573
0.20	0.30	0.06	0.00	0.76327076	0.95828955	0.75934971	0.92598684	0.99171602	0.89855185
0.20	0.30	0.06	0.05	0.05174693	0.08927983	0.04083465	0.11683139	0.14940734	0.08735094
0.20	0.30	0.06	0.10	0.34834492	0.32283193	0.02808240	0.46903339	0.44720242	0.04896164
0.20	0.30	0.06	0.15	0.92690634	0.91270220	0.29749798	0.95916026	0.95156706	0.38699271
0.20	0.30	0.06	0.20	0.99997152	0.99991169	0.99973256	0.99998465	0.99998465	0.99985144
0.20	0.40	0.08	0.00	0.96185513	0.99513668	0.94545236	0.99302586	0.99897828	0.98227624
0.20	0.40	0.08	0.05	0.16957000	0.21670892	0.15642100	0.28676277	0.31182115	0.26152425
0.20	0.40	0.08	0.10	0.11918422	0.12024812	0.03458561	0.20227104	0.19638999	0.06680915
0.20	0.40	0.08	0.15	0.72882373	0.72311962	0.39887014	0.82675840	0.81617066	0.51063050
0.20	0.40	0.08	0.20	0.99967435	0.99977047	0.99855459	0.99989504	0.99990415	0.99955412
0.20	0.50	0.10	0.00	0.99645390	0.99909569	0.98974748	0.99932497	0.99975620	0.99673703
0.20	0.50	0.10	0.05	0.42838704	0.45447294	0.33142509	0.57004444	0.57481128	0.45931545
0.20	0.50	0.10	0.10	0.05075823	0.05991591	0.02882595	0.10699329	0.11012947	0.06343510
0.20	0.50	0.10	0.15	0.42838704	0.45447294	0.32228181	0.57004444	0.57481128	0.45185755
0.20	0.50	0.10	0.20	0.99645390	0.99909569	0.98751417	0.99932497	0.99975620	0.99593942
0.20	0.60	0.12	0.00	0.99967435	0.99977047	0.99861866	0.99989504	0.99990415	0.99956938
0.20	0.60	0.12	0.05	0.72882373	0.72311962	0.40255752	0.82675840	0.81617066	0.51275963
0.20	0.60	0.12	0.10	0.11918422	0.12024812	0.03578909	0.20227104	0.19638999	0.06814086
0.20	0.60	0.12	0.15	0.16957000	0.21670892	0.15452206	0.28676277	0.31182115	0.25951059
0.20	0.60	0.12	0.20	0.96185513	0.99513668	0.94279179	0.99302586	0.99897828	0.98100931
0.20	0.70	0.14	0.00	0.99997152	0.99991169	0.99973256	0.99998465	0.99998465	0.99985144
0.20	0.70	0.14	0.05	0.92690634	0.91270220	0.29755372	0.95916026	0.95156706	0.38701412
0.20	0.70	0.14	0.10	0.34834492	0.32283193	0.02820680	0.46903339	0.44720242	0.04906309
0.20	0.70	0.14	0.15	0.05174693	0.08927983	0.04081340	0.11683139	0.14940734	0.08732419
0.20	0.70	0.14	0.20	0.76327076	0.95828955	0.75917600	0.92598684	0.99171602	0.89844846
0.20	0.80	0.16	0.00	0.99998573	0.99998573	0.99998573	0.99998573	0.99998573	0.99998573
0.20	0.80	0.16	0.05	0.99399003	0.98952707	0.28697455	0.99675405	0.99498856	0.36293700
0.20	0.80	0.16	0.10	0.71938752	0.67156313	0.02388155	0.79901417	0.78154860	0.03803751
0.20	0.80	0.16	0.15	0.08618949	0.08439153	0.00368154	0.14688081	0.16224171	0.01142589
0.20	0.80	0.16	0.20	0.29365989	0.70261136	0.29358031	0.57583625	0.88623181	0.51285569
0.30	0.30	0.09	0.00	0.98772794	0.99942988	0.98737637	0.99849333	0.99995143	0.99743632
0.30	0.30	0.09	0.05	0.25277853	0.29696610	0.25898473	0.39260773	0.41645963	0.38949827
0.30	0.30	0.09	0.10	0.06761925	0.06848270	0.01239139	0.13129569	0.13158212	0.02894315
0.30	0.30	0.09	0.15	0.51967352	0.50540058	0.04749813	0.64766971	0.63927119	0.07873487
0.30	0.30	0.09	0.20	0.95836556	0.95354660	0.28919237	0.97992503	0.97828304	0.37600027
0.30	0.30	0.09	0.25	0.99987703	0.99979794	0.74549205	0.99995311	0.99993796	0.82041704
0.30	0.30	0.09	0.30	0.99999998	0.99999998	0.99999998	0.99999998	0.99999998	0.99999998
0.30	0.40	0.12	0.00	0.99980433	0.99999102	0.99960351	0.99998682	0.99999870	0.99993077
0.30	0.40	0.12	0.05	0.61879414	0.65441930	0.61324788	0.75299161	0.76055257	0.73933905
0.30	0.40	0.12	0.10	0.09040282	0.10480445	0.08451972	0.16809967	0.17314020	0.15062991
0.30	0.40	0.12	0.15	0.16205047	0.16229478	0.03964002	0.25654979	0.25304696	0.07206223
0.30	0.40	0.12	0.20	0.72005815	0.71788882	0.30331780	0.81801053	0.81334925	0.39961276
0.30	0.40	0.12	0.25	0.99101311	0.99041403	0.76542805	0.99641289	0.99598160	0.83317005
0.30	0.40	0.12	0.30	0.99999992	0.99999977	0.99999903	0.99999998	0.99999998	0.99999965
0.30	0.50	0.15	0.00	0.99999734	0.99999912	0.99998582	0.99999957	0.99999972	0.99999702
0.30	0.50	0.15	0.05	0.90340127	0.91022249	0.81673790	0.95011811	0.94980924	0.88553177
0.30	0.50	0.15	0.10	0.34074677	0.35386130	0.25117563	0.46767223	0.46787286	0.35997241
0.30	0.50	0.15	0.15	0.05184249	0.05637951	0.03107693	0.10478651	0.10546837	0.06567413
0.30	0.50	0.15	0.20	0.34074677	0.35386130	0.24078335	0.46767223	0.46787286	0.34931519
0.30	0.50	0.15	0.25	0.90340127	0.91022249	0.80966264	0.95011811	0.94980924	0.88202943
0.30	0.50	0.15	0.30	0.99999734	0.99999912	0.99998072	0.99999957	0.99999972	0.99999621
0.40	0.40	0.16	0.00	0.99999994	1.00000000	0.99999970	1.00000000	1.00000000	0.99999996
0.40	0.40	0.16	0.05	0.93479351	0.94349702	0.92556202	0.96944700	0.97012566	0.96337185
0.40	0.40	0.16	0.10	0.42723692	0.45030246	0.40733539	0.55953342	0.56358048	0.54412155
0.40	0.40	0.16	0.15	0.06085765	0.06688541	0.04761723	0.11812448	0.11980765	0.09428302
0.40	0.40	0.16	0.20	0.22235603	0.22289884	0.07923961	0.33057667	0.33028956	0.13163323
0.40	0.40	0.16	0.25	0.76775865	0.76723805	0.40171840	0.85529161	0.85487607	0.50270645
0.40	0.40	0.16	0.30	0.99192764	0.99174238	0.78509445	0.99694625	0.99689624	0.84321822
0.40	0.40	0.16	0.35	0.99999663	0.99999579	0.97046321	0.99999927	0.99999915	0.98330244
0.40	0.40	0.16	0.40	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.40	0.50	0.20	0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.40	0.50	0.20	0.05	0.99775589	0.99788436	0.97109414	0.99923862	0.99923777	0.98365913
0.40	0.50	0.20	0.10	0.84871917	0.85217550	0.71881883	0.91119960	0.91176920	0.81052277
0.40	0.50	0.20	0.15	0.31119746	0.31650449	0.22094970	0.42902892	0.43065632	0.33182362
0.40	0.50	0.20	0.20	0.05417598	0.05637674	0.03297160	0.10607027	0.10701898	0.07171976
0.40	0.50	0.20	0.25	0.31119746	0.31650449	0.21525593	0.42902892	0.43065632	0.31884723
0.40	0.50	0.20	0.30	0.84871917	0.85217550	0.71355579	0.91119960	0.91176920	0.80398110
0.40	0.50	0.20	0.35	0.99775589	0.99788436	0.97081918	0.99923862	0.99923777	0.98352235
0.40	0.50	0.20	0.40	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.50	0.50	0.25	0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.50	0.50	0.25	0.05	0.99999953	0.99999952	0.99797476	0.99999993	0.99999993	0.99900429
0.50	0.50	0.25	0.10	0.99613308	0.99614584	0.95714086	0.99866202	0.99866739	0.97516784
0.50	0.50	0.25	0.15	0.83512093	0.83576594	0.69348811	0.90298250	0.90336634	0.79665450
0.50	0.50	0.25	0.20	0.30824130	0.30976140	0.21295546	0.42652823	0.42776951	0.33110582
0.50	0.50	0.25	0.25	0.05701837	0.05790733	0.03390766	0.11034443	0.11124826	0.07638645
0.50	0.50	0.25	0.30	0.30824130	0.30976140	0.21099504	0.42652823	0.42776951	0.31726208
0.50	0.50	0.25	0.35	0.83512093	0.83576594	0.69194926	0.90298250	0.90336634	0.78963817
0.50	0.50	0.25	0.40	0.99613308	0.99614584	0.95706617	0.99866202	0.99866739	0.97496917
0.50	0.50	0.25	0.45	0.99999953	0.99999952	0.99797475	0.99999993	0.99999993	0.99900428
0.50	0.50	0.25	0.50	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000

Table 4. Power for n = 100.

P1.	P.1	IP11	P11	UPX	UPL	UPE	UPXU	UPLU	UPEU
0.20	0.20	0.04	0.00	0.92061224	0.99759854	0.94802817	0.99001592	0.99986177	0.98987939
0.20	0.20	0.04	0.05	0.11270129	0.10000841	0.00530322	0.18280474	0.16879634	0.01457428
0.20	0.20	0.04	0.10	0.93542428	0.91947557	0.00199941	0.96246799	0.95385821	0.00441052
0.20	0.20	0.04	0.15	0.99997887	0.99995811	0.17164154	0.99999151	0.99998305	0.25980322
0.20	0.20	0.04	0.20	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.20	0.30	0.06	0.00	0.99920114	0.99999042	0.99943030	0.99996493	0.99999945	0.99994652
0.20	0.30	0.06	0.05	0.07180384	0.09133831	0.07947201	0.14011785	0.15802364	0.15159690
0.20	0.30	0.06	0.10	0.58201792	0.56085680	0.00873339	0.68997833	0.67493276	0.01686317
0.20	0.30	0.06	0.15	0.99731338	0.99659082	0.29277538	0.99885453	0.99858167	0.38831271
0.20	0.30	0.06	0.20	1.00000000	1.00000000	0.99999999	1.00000000	1.00000000	1.00000000
0.20	0.40	0.08	0.00	0.99999311	0.99999970	0.99998264	0.99999959	0.99999996	0.99999734
0.20	0.40	0.08	0.05	0.33109425	0.35343472	0.35301516	0.46012486	0.48166506	0.49506351
0.20	0.40	0.08	0.10	0.18309080	0.17938546	0.03118944	0.27250529	0.27181636	0.05812839
0.20	0.40	0.08	0.15	0.95158980	0.94967010	0.55734562	0.97446778	0.97401994	0.65851821
0.20	0.40	0.08	0.20	0.99999999	0.99999999	0.99999994	1.00000000	1.00000000	0.99999999
0.20	0.50	0.10	0.00	0.99999985	0.99999996	0.99999901	0.99999998	0.99999999	0.99999986
0.20	0.50	0.10	0.05	0.72694151	0.73201402	0.63766625	0.82176337	0.82617964	0.74797821
0.20	0.50	0.10	0.10	0.05140943	0.05317740	0.03433696	0.10098445	0.10454883	0.07273675
0.20	0.50	0.10	0.15	0.72694151	0.73201402	0.63263667	0.82176337	0.82617964	0.74262992
0.20	0.50	0.10	0.20	0.99999985	0.99999996	0.99999919	0.99999998	0.99999999	0.99999983
0.20	0.60	0.12	0.00	0.99999999	0.99999999	0.99999994	1.00000000	1.00000000	0.99999999
0.20	0.60	0.12	0.05	0.95158980	0.94967010	0.55755248	0.97446778	0.97401994	0.65867656
0.20	0.60	0.12	0.10	0.18309080	0.17938546	0.03156161	0.27250529	0.27181636	0.05878697
0.20	0.60	0.12	0.15	0.33109425	0.35343472	0.35237097	0.46012486	0.48166506	0.49413981
0.20	0.60	0.12	0.20	0.99999310	0.99999970	0.99998371	0.99999959	0.99999996	0.99999711
0.20	0.70	0.14	0.00	1.00000000	1.00000000	0.99999999	1.00000000	1.00000000	1.00000000
0.20	0.70	0.14	0.05	0.99731338	0.99659082	0.29277538	0.99885453	0.99858167	0.38831271
0.20	0.70	0.14	0.10	0.58201792	0.56085680	0.00873422	0.68997833	0.67493276	0.01686418
0.20	0.70	0.14	0.15	0.07180384	0.09133831	0.07947175	0.14011785	0.15802364	0.15159639
0.20	0.70	0.14	0.20	0.99920114	0.99999042	0.99943032	0.99996493	0.99999945	0.99994651
0.20	0.80	0.16	0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.20	0.80	0.16	0.05	0.99997887	0.99995811	0.17164154	0.99999151	0.99998305	0.25980322
0.20	0.80	0.16	0.10	0.93542428	0.91947557	0.00199941	0.96246799	0.95385821	0.00441052
0.20	0.80	0.16	0.15	0.11270129	0.10000841	0.00530322	0.18280474	0.16879634	0.01457428
0.20	0.80	0.16	0.20	0.92061224	0.99759854	0.94802817	0.99001592	0.99986176	0.98987939
0.30	0.30	0.09	0.00	0.99999995	1.00000000	0.99999998	1.00000000	1.00000000	1.00000000
0.30	0.30	0.09	0.05	0.48450148	0.51816057	0.53490374	0.62571179	0.64366140	0.67985029
0.30	0.30	0.09	0.10	0.08038383	0.07975781	0.00992395	0.14310123	0.14158184	0.02453158
0.30	0.30	0.09	0.15	0.80377933	0.79807702	0.01376064	0.87633868	0.87217423	0.02462023
0.30	0.30	0.09	0.20	0.99937704	0.99927789	0.23451885	0.99978052	0.99974622	0.31485177
0.30	0.30	0.09	0.25	1.00000000	0.99999999	0.83418609	1.00000000	1.00000000	0.89739007
0.30	0.30	0.09	0.30	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.30	0.40	0.12	0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.30	0.40	0.12	0.05	0.90960444	0.91552441	0.92360015	0.95305273	0.95573208	0.96396312
0.30	0.40	0.12	0.10	0.14135188	0.14833667	0.16086394	0.22872473	0.23564118	0.26344082
0.30	0.40	0.12	0.15	0.27120280	0.26957011	0.03401388	0.38141696	0.37877615	0.05904493
0.30	0.40	0.12	0.20	0.94878751	0.94784224	0.37852676	0.97359995	0.97308586	0.46752622
0.30	0.40	0.12	0.25	0.99998383	0.99998192	0.89715287	0.99999568	0.99999529	0.93335163
0.30	0.40	0.12	0.30	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.30	0.50	0.15	0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.30	0.50	0.15	0.05	0.99661886	0.99666736	0.97657572	0.99875139	0.99877749	0.98753331
0.30	0.50	0.15	0.10	0.59753094	0.59911440	0.48331507	0.71134438	0.71305779	0.60504181
0.30	0.50	0.15	0.15	0.05169135	0.05225425	0.03501496	0.10118867	0.10220868	0.07527676
0.30	0.50	0.15	0.20	0.59753094	0.59911440	0.47829431	0.71134438	0.71305779	0.59930672
0.30	0.50	0.15	0.25	0.99661886	0.99666736	0.97639361	0.99875139	0.99877749	0.98743099
0.30	0.50	0.15	0.30	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.40	0.40	0.16	0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.40	0.40	0.16	0.05	0.99885194	0.99890173	0.99901758	0.99964049	0.99965689	0.99971349
0.40	0.40	0.16	0.10	0.72112322	0.72425128	0.74698097	0.81801815	0.82123231	0.84568992
0.40	0.40	0.16	0.15	0.07054650	0.07164128	0.07369861	0.13000743	0.13208892	0.13653093
0.40	0.40	0.16	0.20	0.38716536	0.38707211	0.07450473	0.51062485	0.50862962	0.11278432
0.40	0.40	0.16	0.25	0.96788920	0.96784182	0.46454757	0.98490115	0.98462033	0.55536888
0.40	0.40	0.16	0.30	0.99998886	0.99998870	0.88875020	0.99999744	0.99999729	0.92456180
0.40	0.40	0.16	0.35	1.00000000	1.00000000	0.99705121	1.00000000	1.00000000	0.99872746
0.40	0.40	0.16	0.40	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.40	0.50	0.20	0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.40	0.50	0.20	0.05	0.99999951	0.99999951	0.99846372	0.99999991	0.99999991	0.99925644
0.40	0.50	0.20	0.10	0.98881514	0.98885947	0.92785703	0.99528152	0.99536023	0.95410360
0.40	0.50	0.20	0.15	0.53975902	0.54034116	0.42192935	0.65693783	0.65929641	0.52952913
0.40	0.50	0.20	0.20	0.05209335	0.05230060	0.03679760	0.10063784	0.10202107	0.07405490
0.40	0.50	0.20	0.25	0.53975902	0.54034116	0.41476906	0.65693783	0.65929641	0.52387489
0.40	0.50	0.20	0.30	0.98881514	0.98885947	0.92721972	0.99528152	0.99536023	0.95382760
0.40	0.50	0.20	0.35	0.99999951	0.99999951	0.99846365	0.99999991	0.99999991	0.99925642
0.40	0.50	0.20	0.40	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.50	0.50	0.25	0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.50	0.50	0.25	0.05	1.00000000	1.00000000	0.99999180	1.00000000	1.00000000	0.99999720
0.50	0.50	0.25	0.10	0.99999808	0.99999808	0.99739210	0.99999955	0.99999956	0.99878065
0.50	0.50	0.25	0.15	0.98560492	0.98560738	0.92665155	0.99316796	0.99328317	0.95470212
0.50	0.50	0.25	0.20	0.52781100	0.52786908	0.42471077	0.63413206	0.63692052	0.52118745
0.50	0.50	0.25	0.25	0.05354792	0.05358333	0.03973707	0.09701016	0.09876556	0.07388305
0.50	0.50	0.25	0.30	0.52781100	0.52786908	0.41283152	0.63413206	0.63692052	0.51211814
0.50	0.50	0.25	0.35	0.98560492	0.98560738	0.92540939	0.99316796	0.99328317	0.95409287
0.50	0.50	0.25	0.40	0.99999808	0.99999808	0.99739177	0.99999955	0.99999956	0.99878055
0.50	0.50	0.25	0.45	1.00000000	1.00000000	0.99999180	1.00000000	1.00000000	0.99999720
0.50	0.50	0.25	0.50	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000

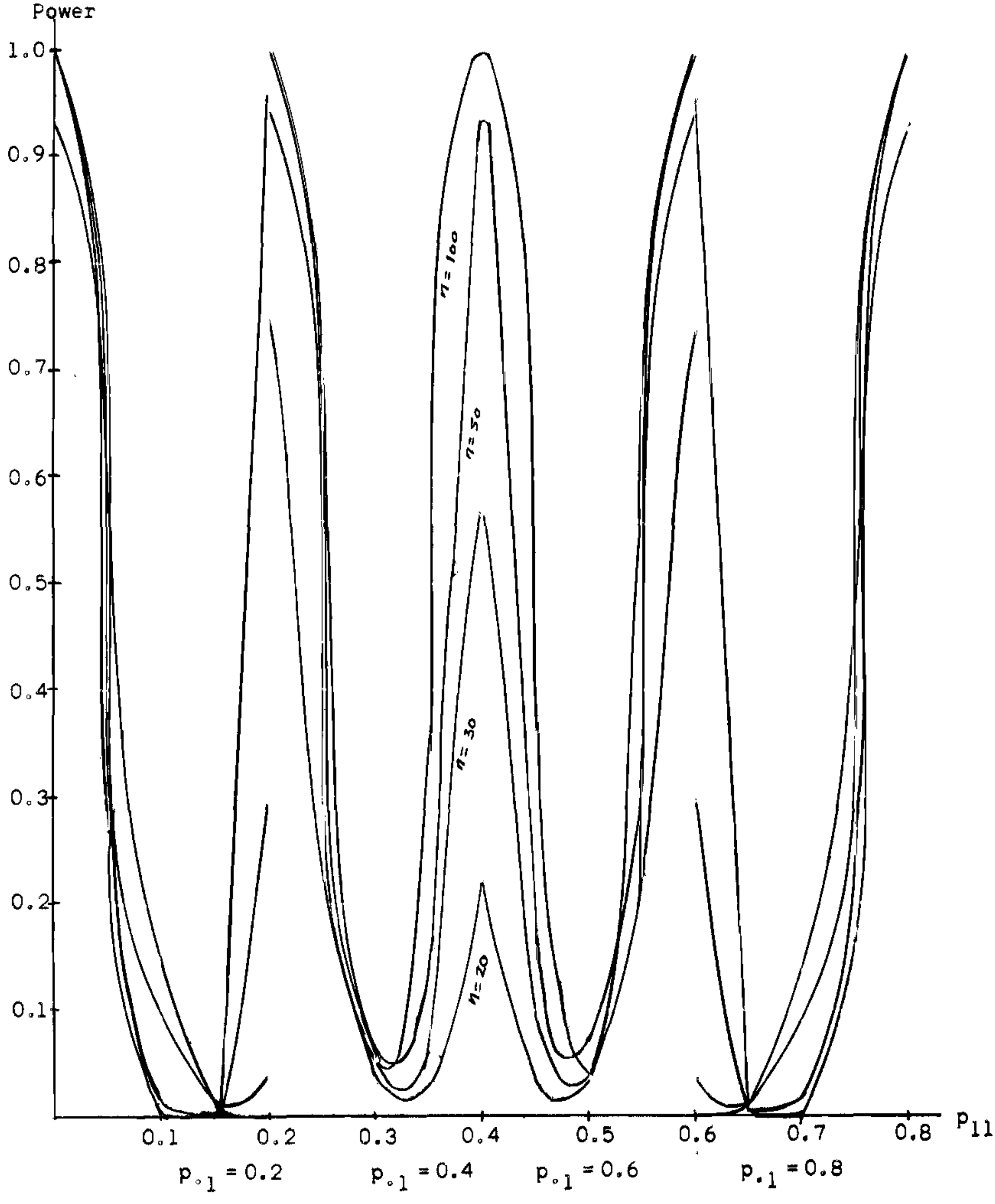

Figure 4. Power Curves for the Exact Test with $\alpha = 0.05$ and $p_{1\cdot} = 0.8$.

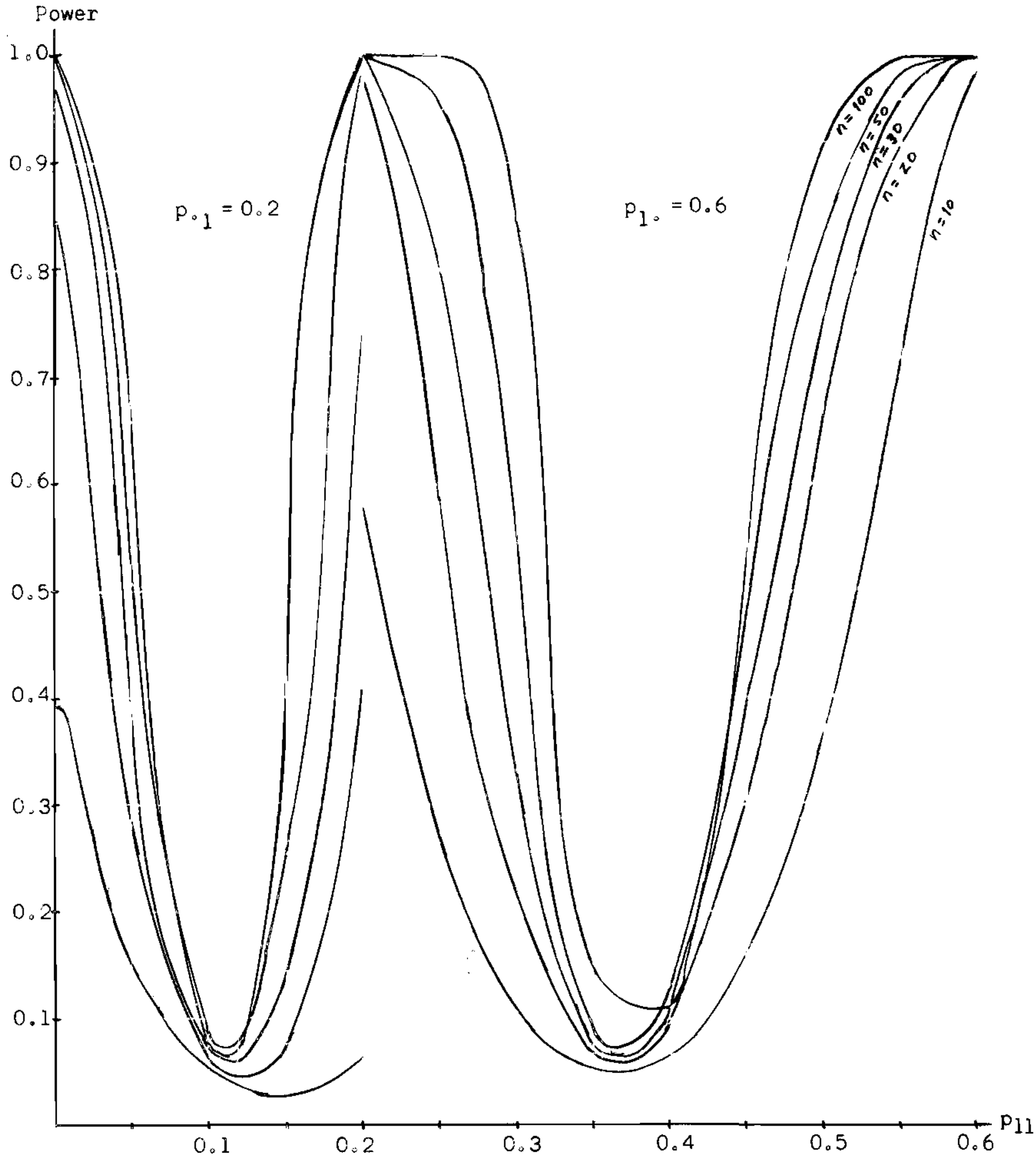

Figure 5. Power Curves for the Exact Test with $\alpha = 0.10$ and $p_{1.} = 0.6$.

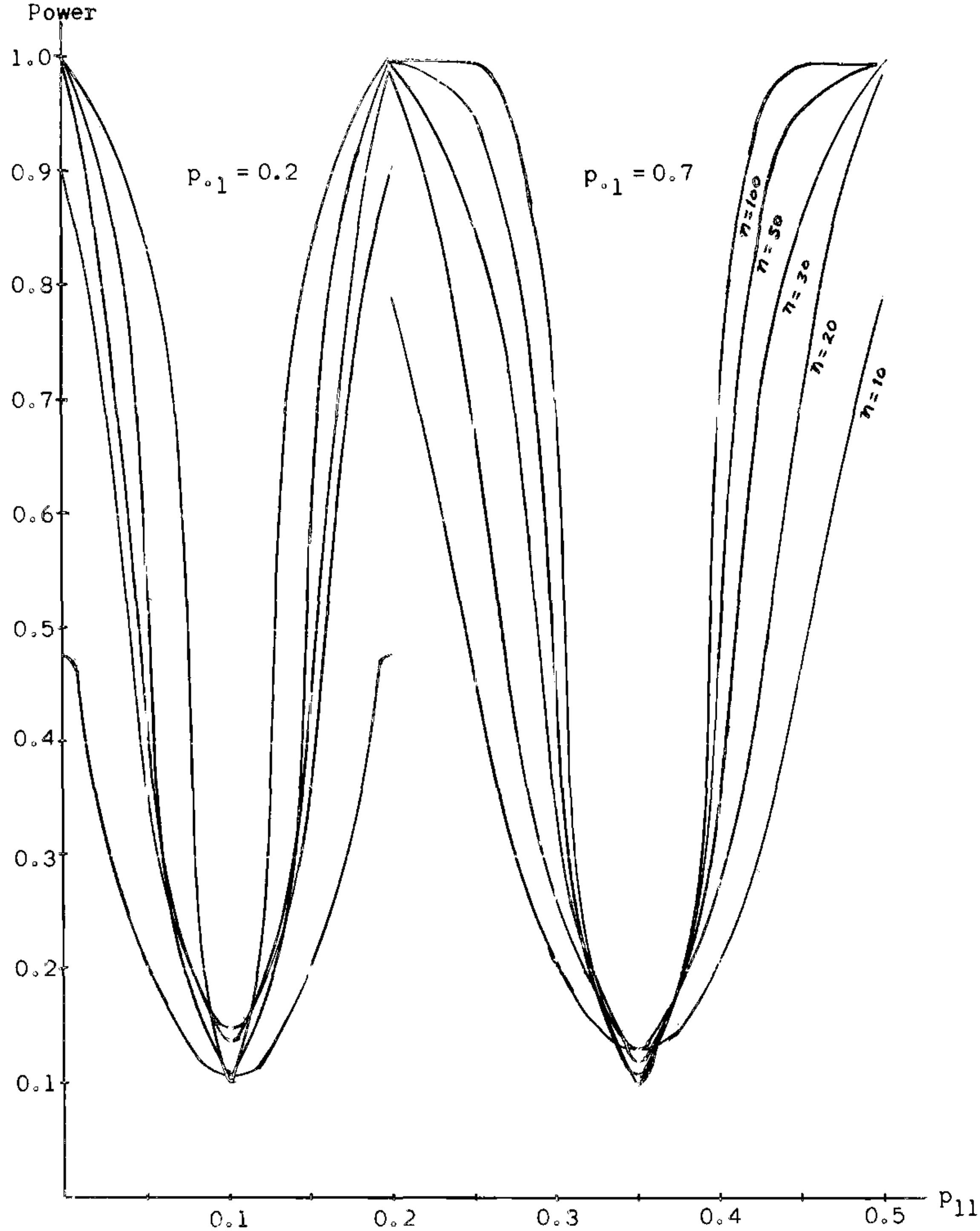

Figure 6. Power Curves for the $-2 \log \lambda$ Statistic Test with $\alpha = 0.10$ and $p_{1.} = 0.5$.

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